Introduction to CAP: Constructive category theory and applications

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University of Siegen

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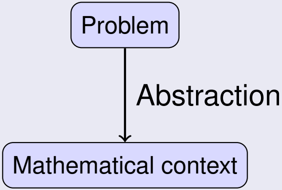
Part I

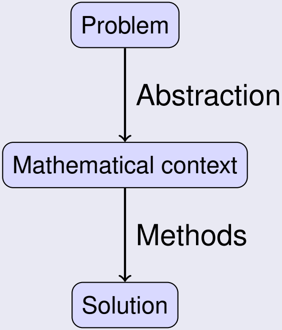
Constructive category theory

Problem

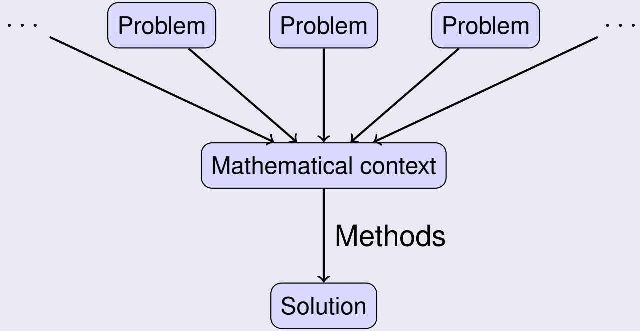
Mathematics



Mathematics



Mathematics



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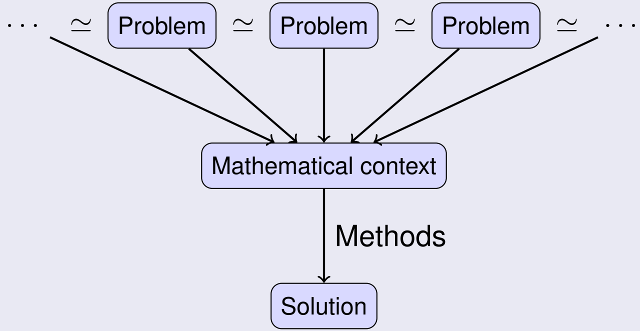
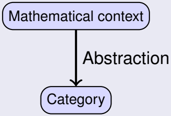


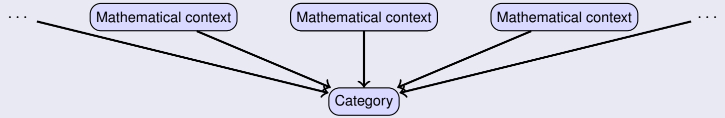
Constructive category theory

Mathematical context

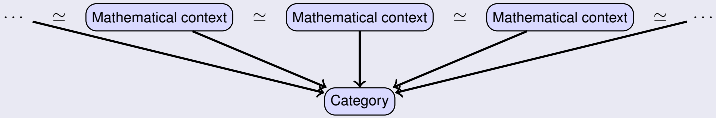
Constructive category theory



Constructive category theory



Constructive category theory



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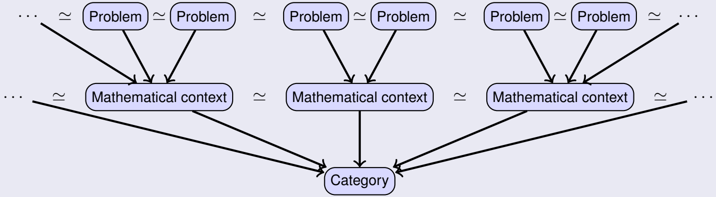

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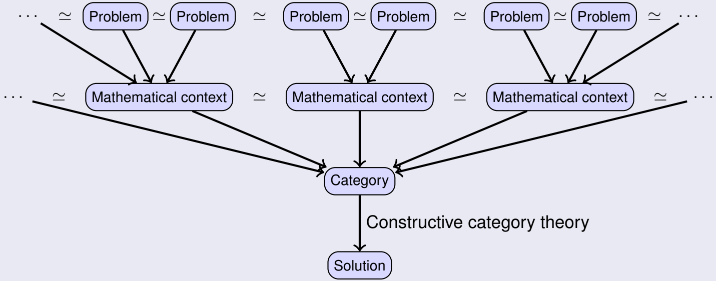

Abstraction of language

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Addition of two numbers:

Data type: `int`

Data type: `float`

Abstraction of language

Addition of two numbers: Assembly

Data type: `int`

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Abstraction of language

Addition of two numbers: Assembly

Data type: `int`

```
addi:
movl   %edi, -4(%rsp)
movl   %esi, -8(%rsp)
movl   -4(%rsp), %esi
addl   -8(%rsp), %esi
movl   %esi, %eax
ret
```

Data type: `float`

Abstraction of language

Addition of two numbers: Assembly

Data type: `int`

```
addi:
movl   %edi, -4(%rsp)
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movl   -4(%rsp), %esi
addl   -8(%rsp), %esi
movl   %esi, %eax
ret
```

Data type: `float`

```
addf:
movss %xmm0, -4(%rsp)
movss %xmm1, -8(%rsp)
movss -4(%rsp), %xmm0
addss -8(%rsp), %xmm0
ret
```

Abstraction of language

Addition of two numbers: C

Data type: `int`

Data type: `float`

Abstraction of language

Addition of two numbers: C

Data type: `int`

```
int addi( int a,
          int b )
{
  return a + b;
}
```

Data type: `float`

Abstraction of language

Addition of two numbers: C

Data type: `int`

```
int addi( int a,
          int b )
{
  return a + b;
}
```

Data type: `float`

```
float addf( float a,
            float b )
{
  return a + b;
}
```

Abstraction of language

Addition of two numbers: GAP or Julia

Data type: `int`

Data type: `float`

Abstraction of language

Addition of two numbers: GAP or Julia

Data type: `int`

```
function( a, b )
    return a + b;
end;
```

Data type: `float`

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High language leads to generic code!

Abstraction of language

Computing the intersection of two subobjects

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Vector spaces

$\langle v_1, v_2 \rangle, \langle w_1, w_2 \rangle \leq V$:

Abstraction of language

Computing the intersection of two subobjects

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Solution of

$$\begin{aligned} x_1 v_1 + x_2 v_2 \\ = y_1 w_1 + y_2 w_2 \end{aligned}$$

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Ideals of $\mathbb{Z}$

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Ideals of $\mathbb{Z}$

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Euclidean algorithm:

$$\langle \operatorname{lcm}(x, y) \rangle$$

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Generic algorithm for both cases?

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Generic algorithm for both cases? **Category theory!**

Category theory as programming language

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CAP - Categories, Algorithms, Programming

CAP implements a
categorical programming language

Categories

Definition

A category $\mathcal{A}$ contains the following data:

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- $\mathrm{Obj}_{\mathcal{A}}$

A B C

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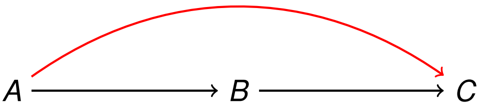

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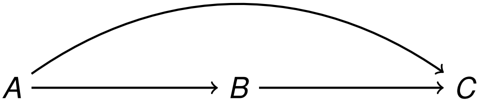

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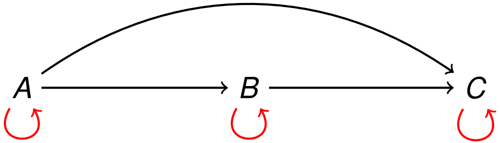

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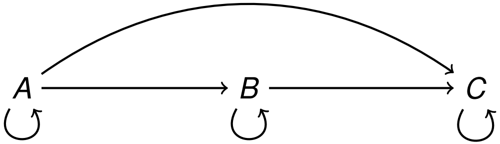

Finite dimensional vector spaces

Let k be a field.

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Example: k-vec

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Example: matrices

- $\mathrm{Obj} := \mathbb{N}_0$

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Example: matrices (computerfriendly model)

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Computable categories

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1 2 1

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Example: $\mathbb{Q}$-vec

$$1 \xrightarrow[\begin{pmatrix} 1 & 2 \end{pmatrix}]{} 2 \xrightarrow[\begin{pmatrix} 3 \\ 4 \end{pmatrix}]{} 1$$

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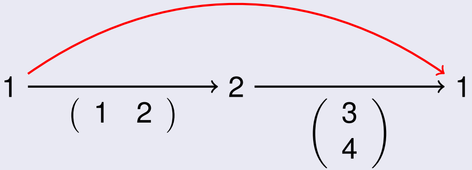

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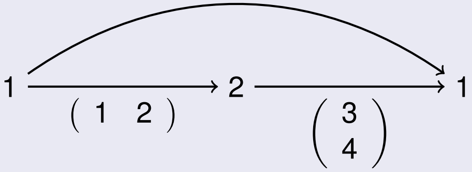

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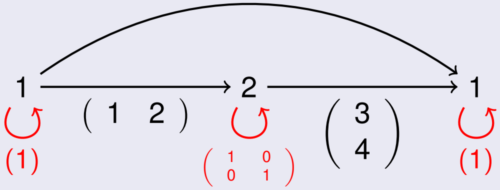

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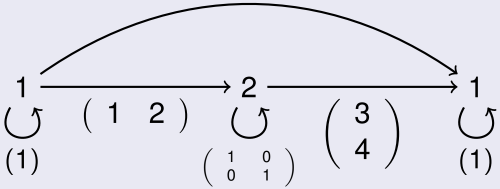

Equivalences

Example

$$\mathrm{Rep}_k(G) \xrightarrow{\quad\quad\quad\quad\sim\quad\quad\quad\quad} \bigoplus_{i \in \mathrm{Irr}(G)} k\text{-vec}$$

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$$V \simeq \bigoplus_{i \in \mathrm{Irr}(G)} a_i V^i$$

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Equivalences

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Example: S_3, irreducible representations: $V^{\mathbb{1}}, V^{\mathrm{sgn}}, V^{\chi}$

$$
\begin{array}{ccc}
V & \xrightarrow{\begin{pmatrix} -102 & 5824 & -96 & 20 & 1444 & 584 \\ 58 & -2366 & 60 & 8 & -590 & -240 \\ 83 & -5366 & 75 & -28 & -1328 & -536 \\ -25 & 1354 & -24 & 3 & 336 & 136 \\ -377 & 17200 & -384 & -28 & 4279 & 1736 \\ 351 & -18877 & 348 & -12 & -4682 & -1893 \end{pmatrix}} & V \\
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V^{\mathbb{1}} \oplus V^{\mathrm{sgn}} \oplus V^{\chi} \oplus V^{\chi} & \xrightarrow[\mathrm{Diag}(-1, 3, \begin{pmatrix} -2 & 4 \\ 1 & -1 \end{pmatrix} \otimes I_2)]{} & V^{\mathbb{1}} \oplus V^{\mathrm{sgn}} \oplus V^{\chi} \oplus V^{\chi}
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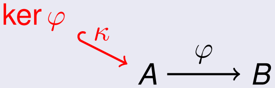

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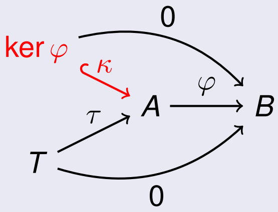

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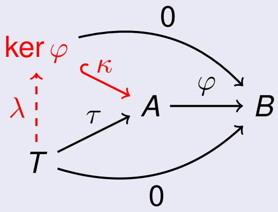

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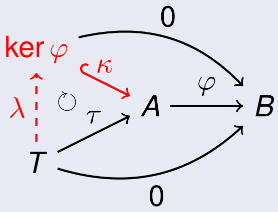

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$\mathrm{Obj} := \mathbb{Z}_{\geq 0}$, $\mathrm{Hom}\,(m, n) := \mathbb{Q}^{m \times n}$

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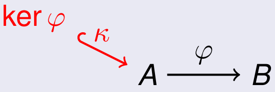

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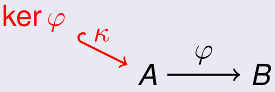

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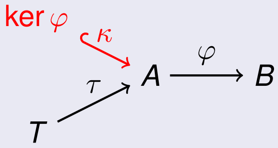

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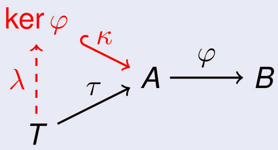

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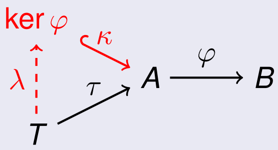

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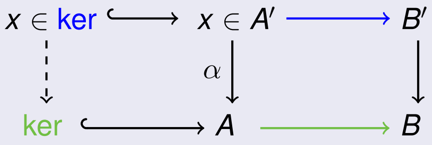

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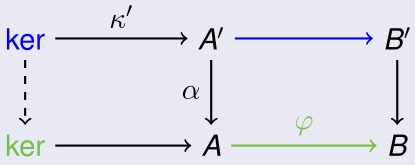

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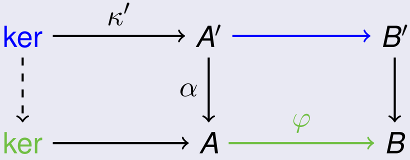

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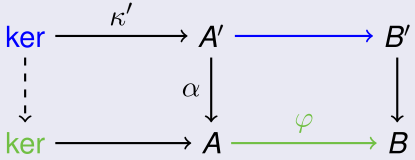

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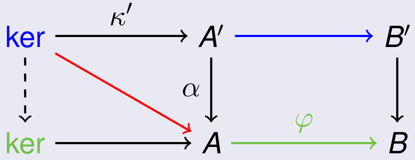

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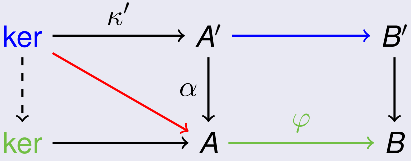

$$\downarrow = \mathrm{KernelLift}(\varphi, \alpha \circ \kappa')$$

Features of CAP

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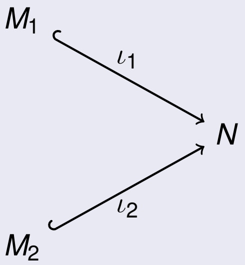

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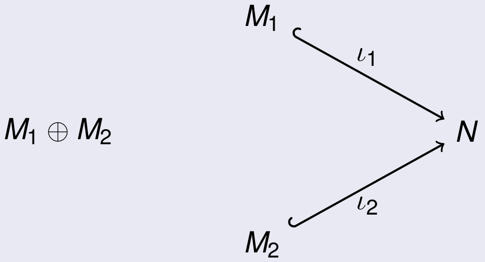

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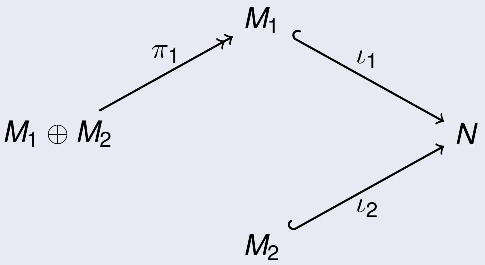

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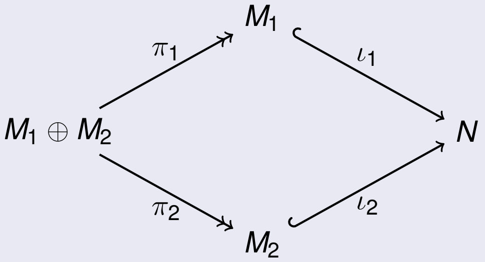

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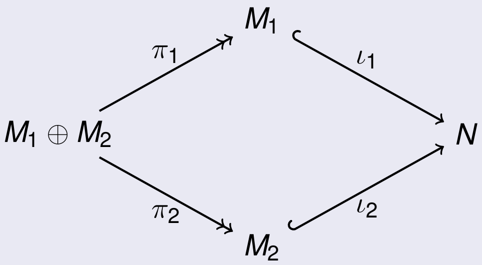

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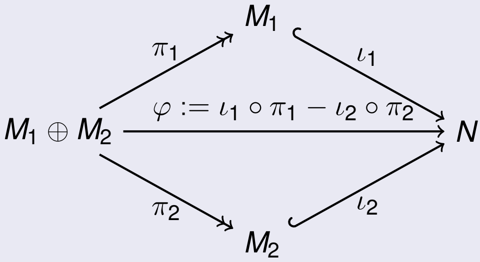

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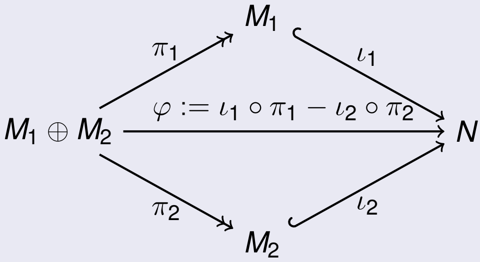

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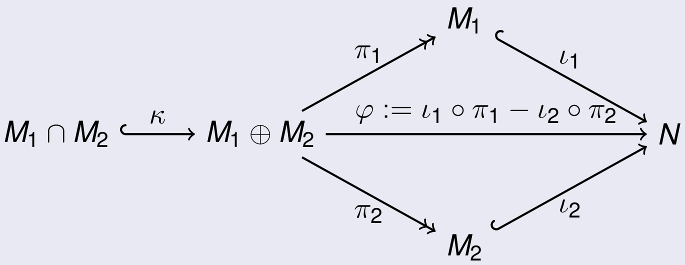

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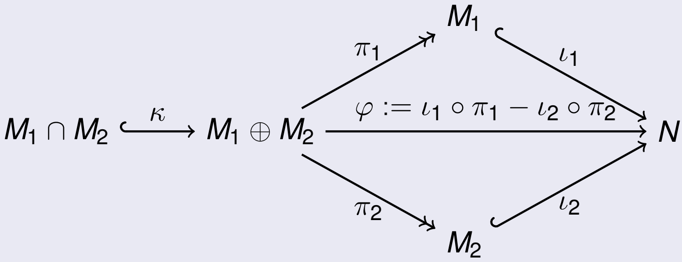

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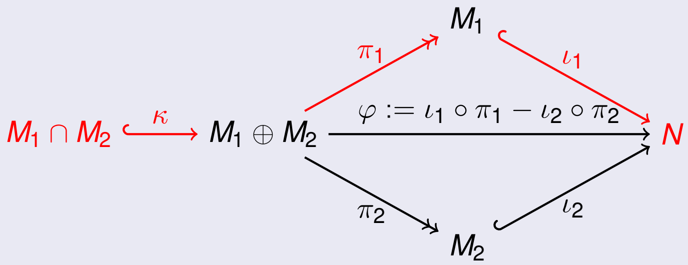

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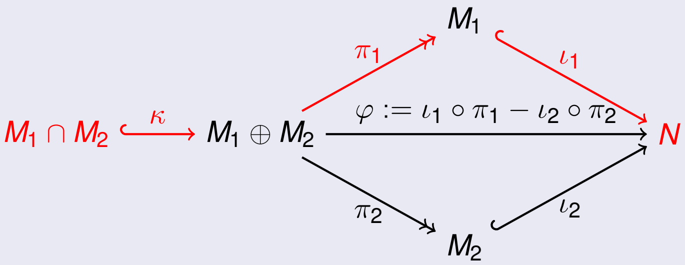

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  return gamma;
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```

Translation to CAP

```
IntersectionOfSubobject := function( iota1, iota2 )
  local M1, M2, pi1, pi2, lambda, phi, kappa, gamma;
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Computing the intersection: $\mathbb{Q}$-vec

Compute the intersection of

$$\begin{array}{ccccc} M_1 & \xhookrightarrow{\iota_1 := \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}} & N & \xhookleftarrow{\iota_2 := \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}} & M_2 \\ \cong & & \cong & & \cong \\ 2 & & 3 & & 2 \end{array}$$

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$$M_1 \simeq 2, \quad N \simeq 3, \quad M_2 \simeq 2$$

```
gap> gamma := IntersectionOfSubobject( iota1, iota2 );
<A morphism in the category of matrices over Q>
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$$M_1 \cong 2, \quad N \cong 3, \quad M_2 \cong 2$$

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gap> gamma := IntersectionOfSubobject( iota1, iota2 );
<A morphism in the category of matrices over Q>

gap> Display( gamma );
[ [  1,  1,  0 ] ]

A morphism in the category of matrices over Q
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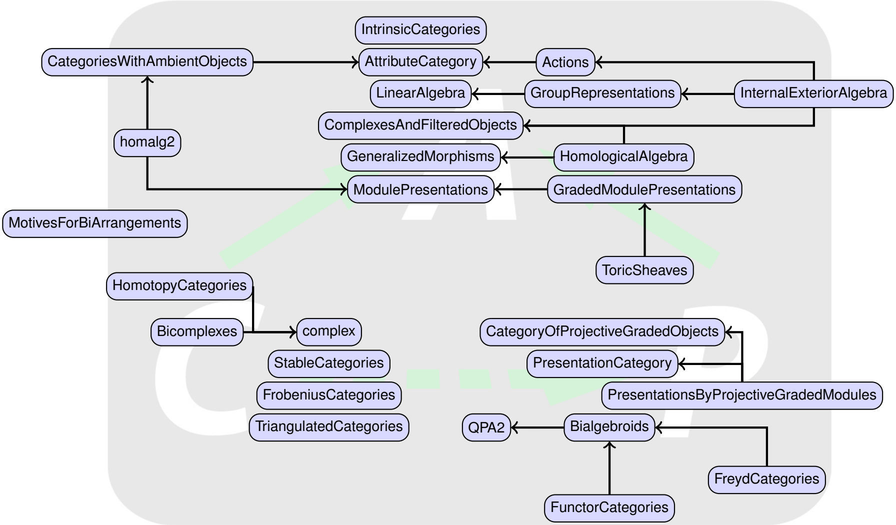
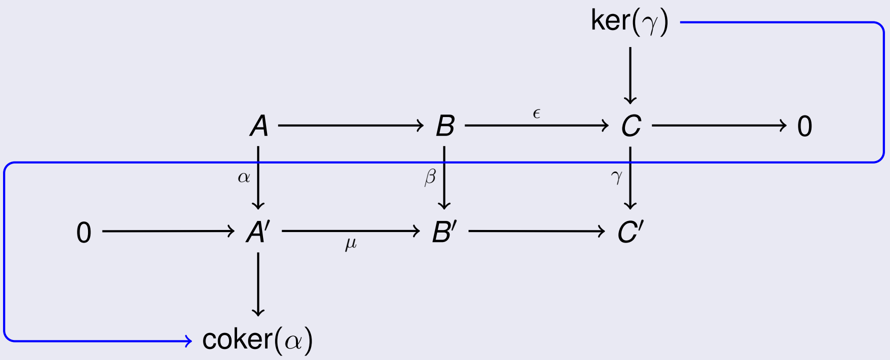

IntrinsicCategories
CategoriesWithAmbientObjects
AttributeCategory
Actions
LinearAlgebra
GroupRepresentations
InternalExteriorAlgebra
ComplexesAndFilteredObjects
homalg2
GeneralizedMorphisms
HomologicalAlgebra
ModulePresentations
GradedModulePresentations
MotivesForBiArrangements
ToricSheaves
HomotopyCategories
Bicomplexes
complex
CategoryOfProjectiveGradedObjects
StableCategories
PresentationCategory
FrobeniusCategories
PresentationsByProjectiveGradedModules
TriangulatedCategories
QPA2
Bialgebroids
FreydCategories
FunctorCategories

$$
\begin{array}{ccccccccc}
 & & & & & & \ker(\gamma) & & \\
 & & & & & & \downarrow & & \\
 & & A & \longrightarrow & B & \xrightarrow{\epsilon} & C & \longrightarrow & 0 \\
 & & {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & {\scriptstyle\gamma}\downarrow & & \\
0 & \longrightarrow & A' & \xrightarrow[\mu]{} & B' & \longrightarrow & C' & & \\
 & & \downarrow & & & & & & \\
 & & \operatorname{coker}(\alpha) & & & & & &
\end{array}
$$

Snake lemma



Part II

Generalized morphisms

1. Classical diagram chases
2. Additive relations
3. Generalized morphisms
4. Applications of generalized morphisms
   - An algorithm for spectral sequences
   - The purity filtration

What are diagram chases?

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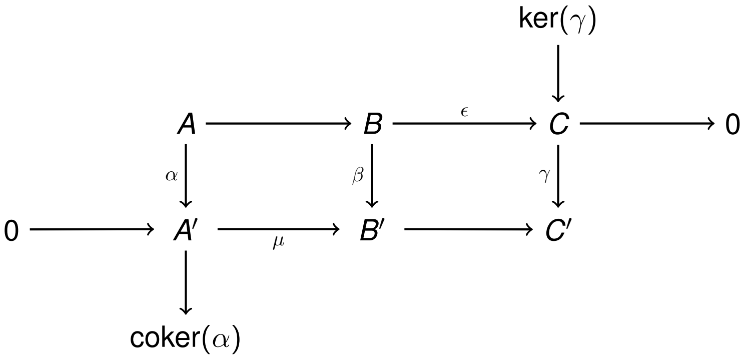
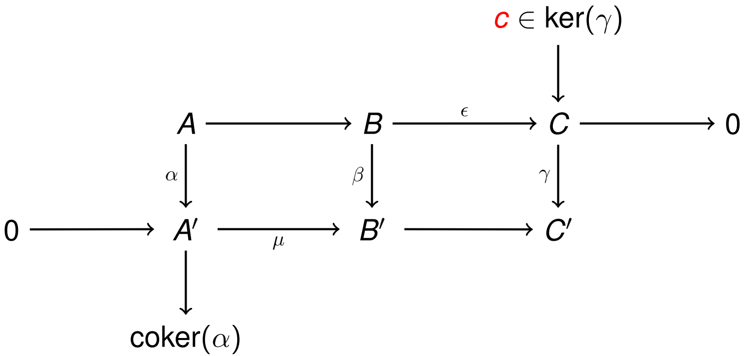
Connecting homomorphism in the snake lemma



Wanted: $\ker(\gamma) \xrightarrow{\partial} \operatorname{coker}(\alpha)$.

Connecting homomorphism in the snake lemma



Start: $c \in \ker(\gamma)$.

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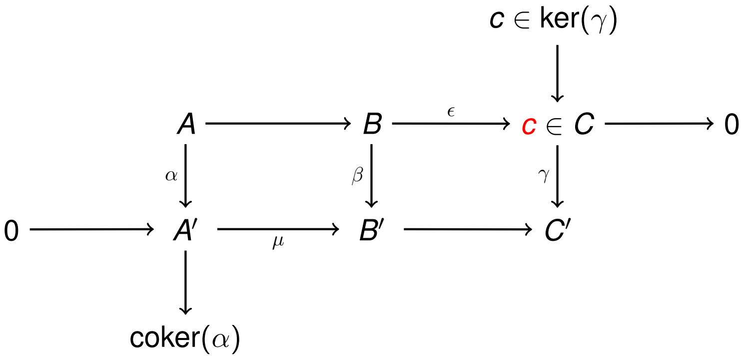
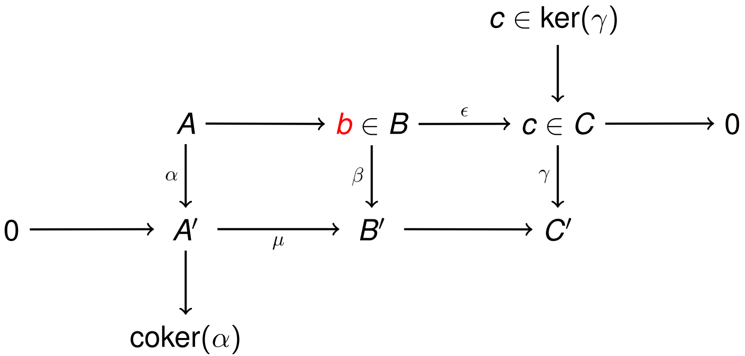


This lies in C.

Connecting homomorphism in the snake lemma



Choose: $b \in \epsilon^{-1}(\{c\})$.

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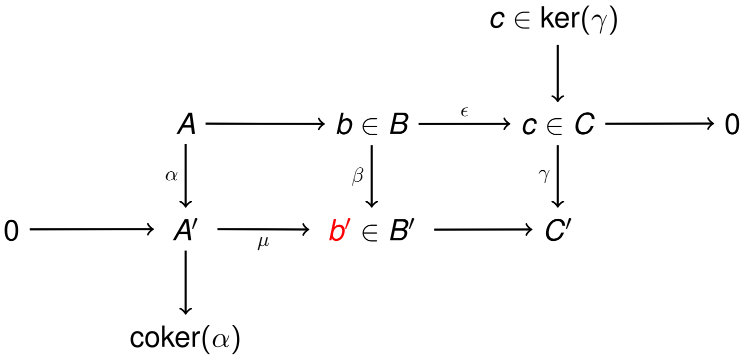
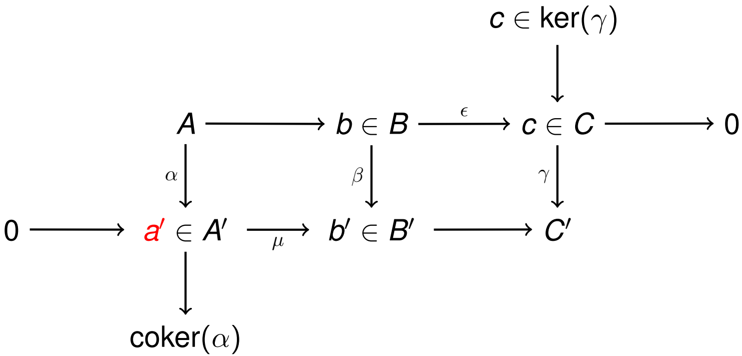


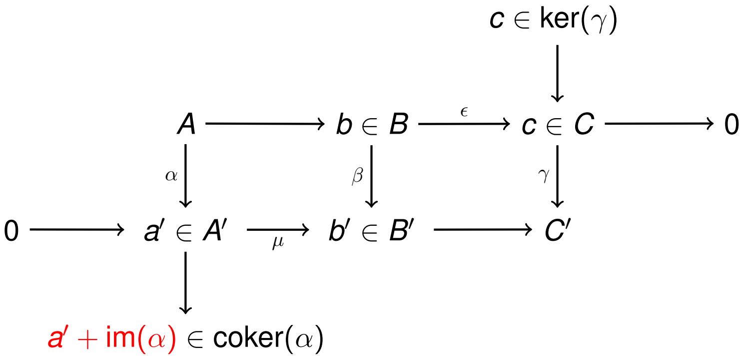
Map: $b \overset{\beta}{\mapsto} b'$.

Connecting homomorphism in the snake lemma



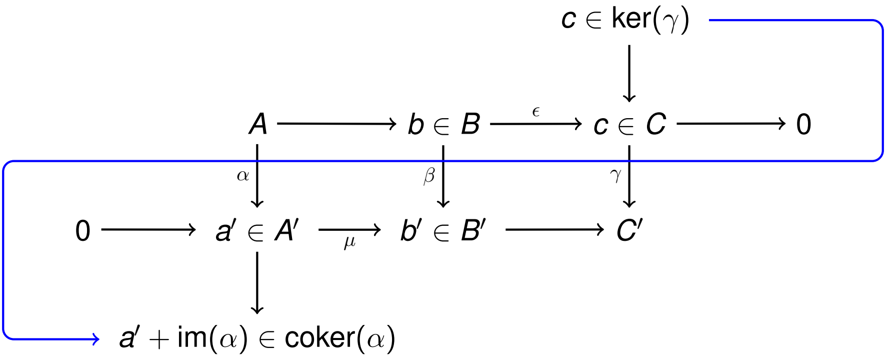
Compute: $a' \in \mu^{-1}(b')$.

Connecting homomorphism in the snake lemma



Map: $a' \mapsto a' + \mathrm{im}(\alpha)$.

Connecting homomorphism in the snake lemma



Result: $c \overset{\partial}{\mapsto} a' + \mathrm{im}(\alpha)$.

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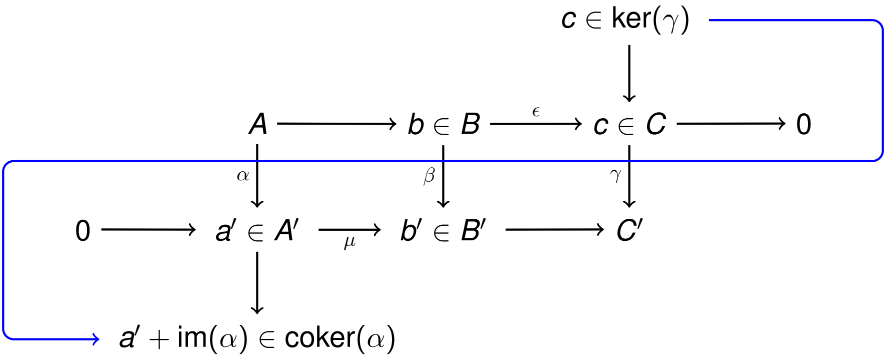

Result: $c \overset{\partial}{\mapsto} a' + \mathrm{im}(\alpha)$. **Context**: modules

Classical solutions: embedding theorems

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Freyd-Mitchell embedding theorem

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Any small abelian category $\mathbf{A}$ admits an exact fully faithful covariant embedding

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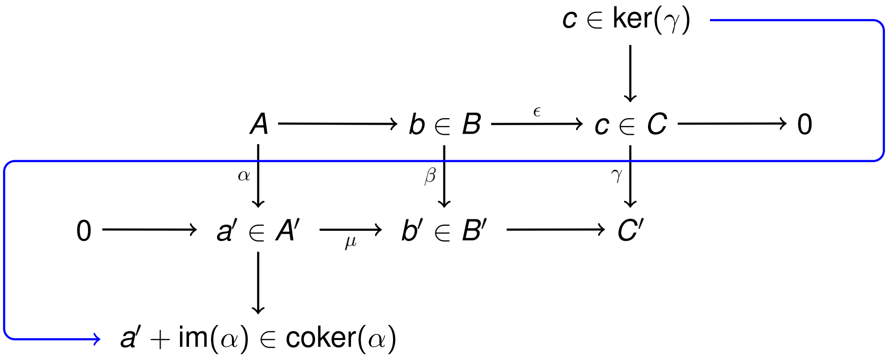
Problem: this isomorphism between Hom-sets is **not constructive.**

Back to the snake lemma



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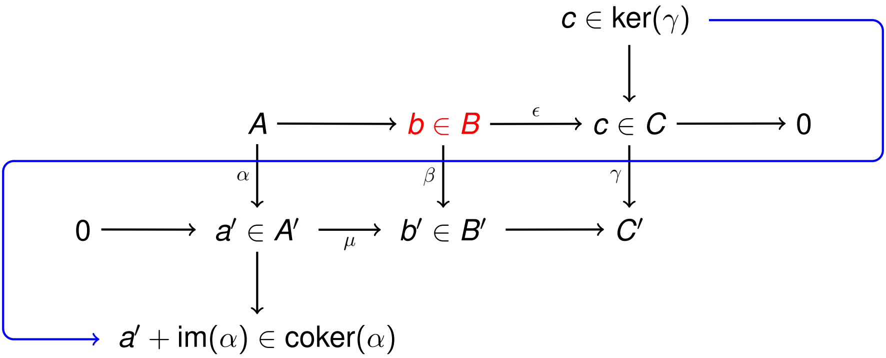
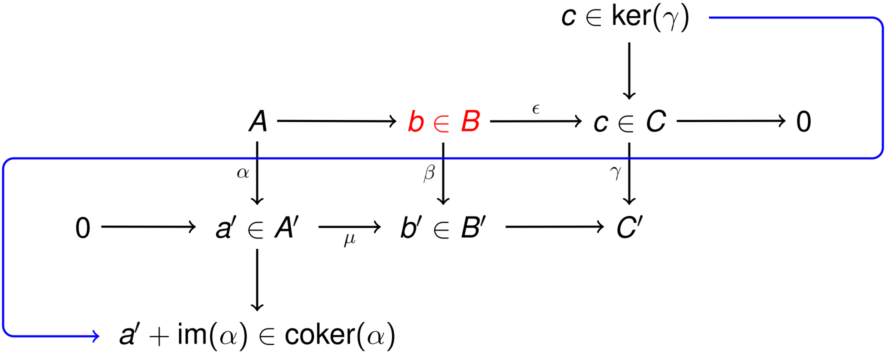


Crucial step: the **uncanonical** choice $b \in \epsilon^{-1}(\{c\})$.

Back to the snake lemma



Make this step canonical: **relations** instead of maps: $c \mapsto \epsilon^{-1}(\{c\})$

Relations

Let A, B be abelian groups.

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Definition

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is a relation from B to A, called **pseudo-inverse of** ϵ.

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Given $f \subseteq A \oplus B$ and $g \subseteq B \oplus C$, define

$$g \circ f := \{(a,c) \in A \oplus C \mid \exists b \in B : (a,b) \in f, (b,c) \in g\}$$

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If f and g correspond to maps, this describes their usual composition.

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Q: When does an additive relation $f \subseteq A \oplus B$ defines an honest map (a group homomorphism)?

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A: When it has a full domain

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$\mathrm{dom}(f) := \{a \in A \mid \exists b \in B : (a,b) \in f\} = A$

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Q: When does an additive relation $f \subseteq A \oplus B$ defines an honest map (a group homomorphism)?

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$\mathrm{dom}(f) := \{a \in A \mid \exists b \in B : (a, b) \in f\} = A$

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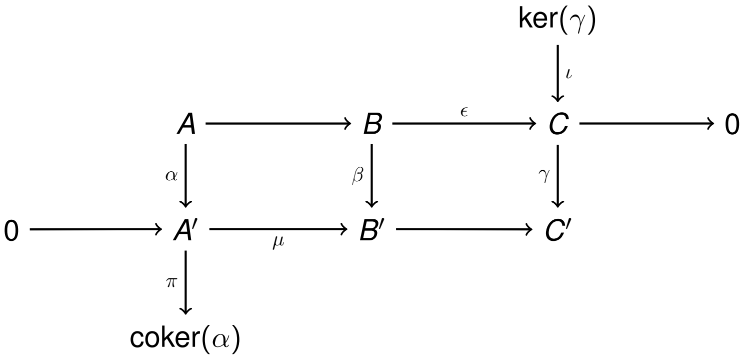
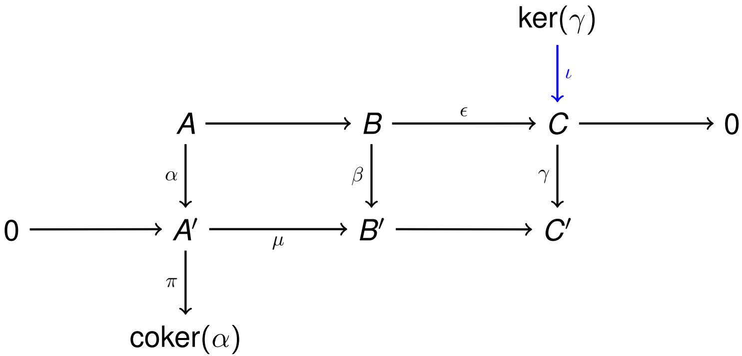
The snake lemma for a last time



Wanted: $\ker(\gamma) \xrightarrow{\partial} \mathrm{coker}(\alpha)$.

The snake lemma for a last time



ι

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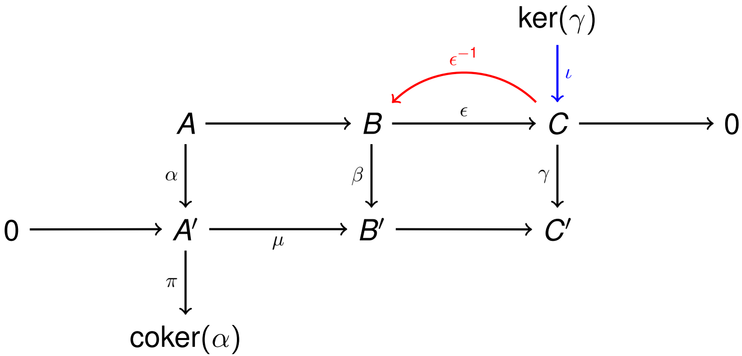

$$\epsilon^{-1} \circ \iota$$

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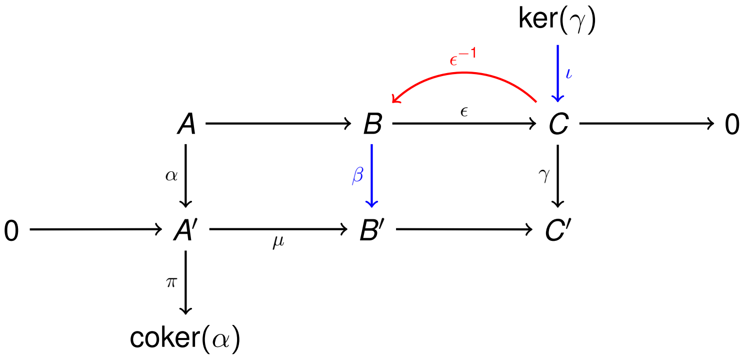

$$\beta \circ \epsilon^{-1} \circ \iota$$

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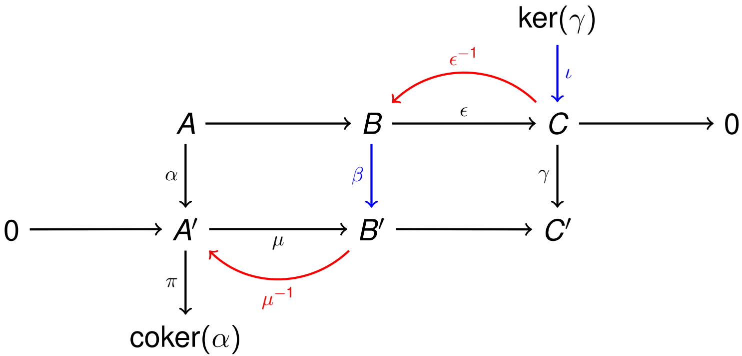

$$\mu^{-1} \circ \beta \circ \epsilon^{-1} \circ \iota$$

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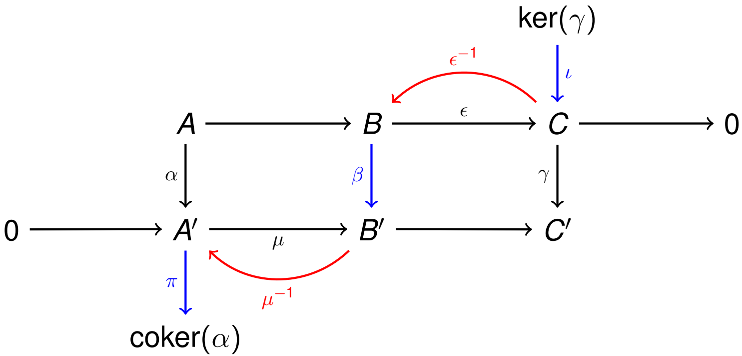

$$\pi \circ \mu^{-1} \circ \beta \circ \epsilon^{-1} \circ \iota$$

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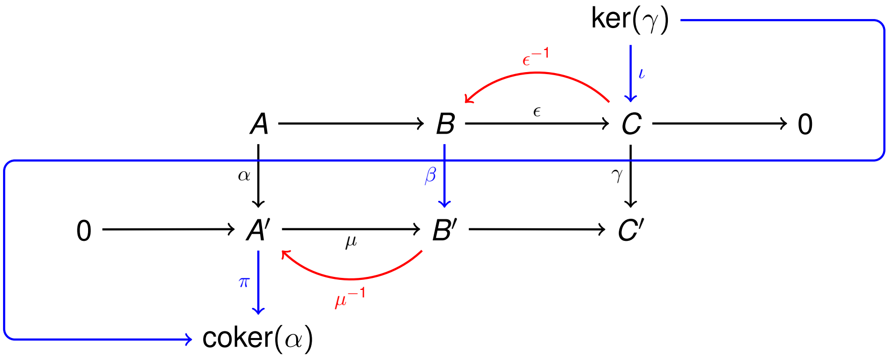

∂ is an honest map given by a composition of relations!

From relations to generalized morphisms

- **Wanted**: a categorical framework for relations.

From relations to generalized morphisms

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- **Solution**: generalized morphisms.

From relations to generalized morphisms

Let A, B be objects in an abelian category $\mathbf{A}$.

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Relation

$$\begin{array}{c} A \oplus B \\ \Big\uparrow \\ D \end{array}$$

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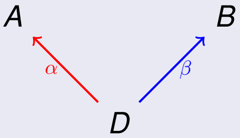

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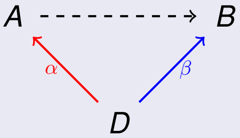

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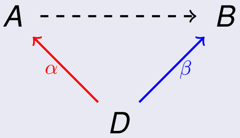
Relation $\rightsquigarrow$ generalized morphism (data structure: span)



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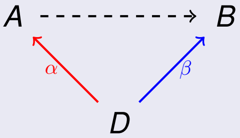

Equality

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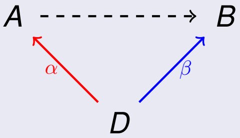

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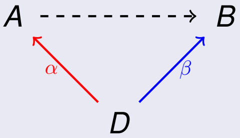

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$$\mathrm{im}\left((\alpha, \beta) : D \to A \oplus B\right) = \mathrm{im}\left((\alpha', \beta') : D' \to A \oplus B\right).$$

Composition of generalized morphisms

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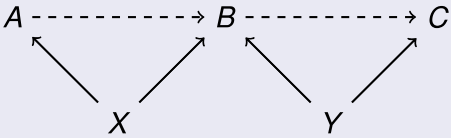

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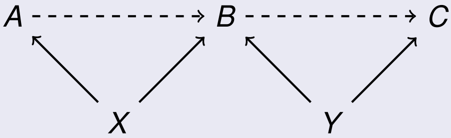

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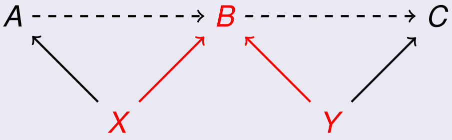

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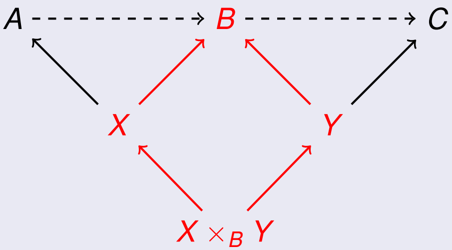

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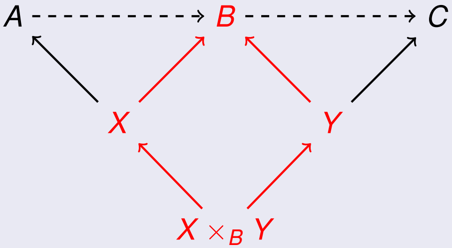

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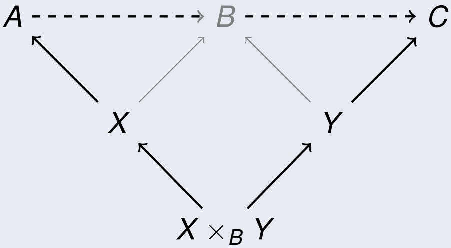

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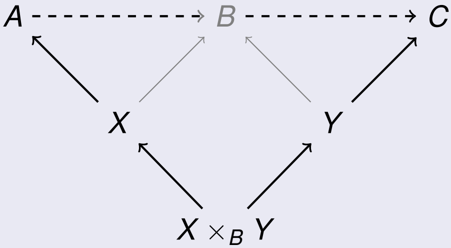

⇝ Category of generalized morphisms G(**A**)

Pseudo-inverses

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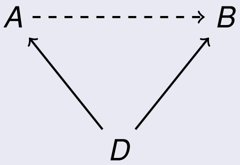

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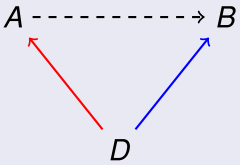

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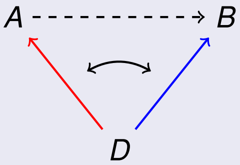

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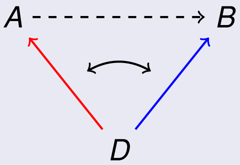

$\rightsquigarrow$

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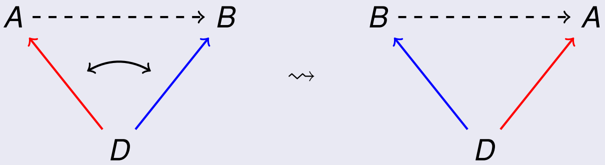

Honest morphisms

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$\mathbf{A}$ embeds into $\mathrm{G}(\mathbf{A})$:

Honest morphisms

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A embeds into G(**A**):

$$A \longrightarrow B$$

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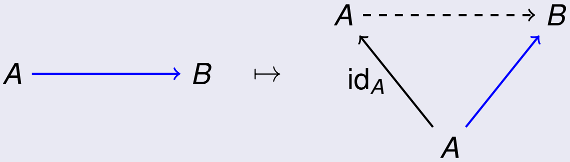

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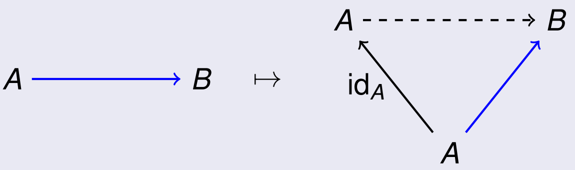

Generalized morphisms with such a representation are called **honest.**

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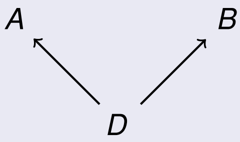

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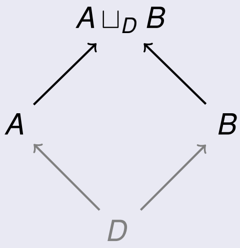

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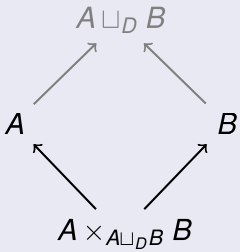

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$$\begin{array}{ccc} A & & B \\ \overset{\sim}{\nwarrow} & & \nearrow \\ & A \times_{A \sqcup_D B} B & \end{array}$$

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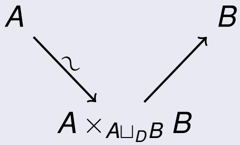

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$$A \longrightarrow B$$

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$$\mathbb{Q} \xleftarrow{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} \mathbb{Q}^2 \xrightarrow{\begin{pmatrix} 2 \\ -2 \end{pmatrix}} \mathbb{Q}$$

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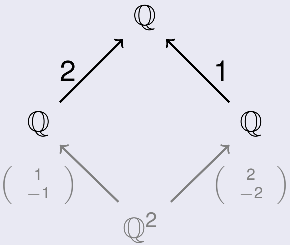

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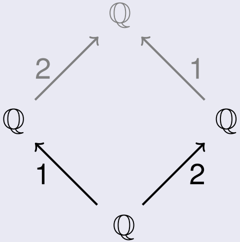

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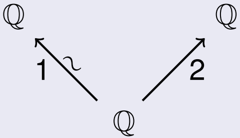

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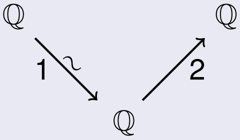

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$$\mathbb{Q} \xrightarrow{\quad 2 \quad} \mathbb{Q}$$

Constructive diagram chases

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Strategy for constructive diagram chases

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1. Compute in $G(\mathbf{A})$ using pseudo-inverses and compositions.

Constructive diagram chases

Strategy for constructive diagram chases

1. Compute in $G(\mathbf{A})$ using pseudo-inverses and compositions.
2. Compute the honest representative of the resulting generalized morphism.

Example: functoriality of homology

Let $(P_\bullet, \partial)$ be a complex in an abelian category $\mathcal{A}$.

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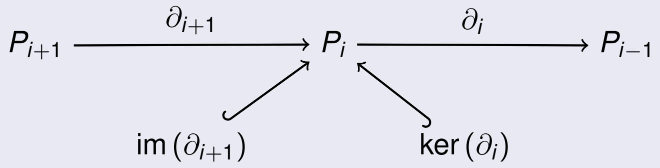

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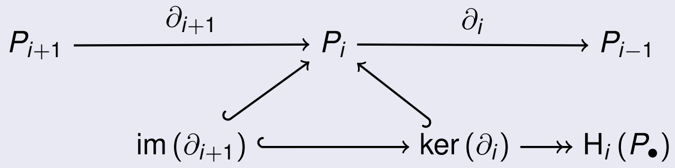

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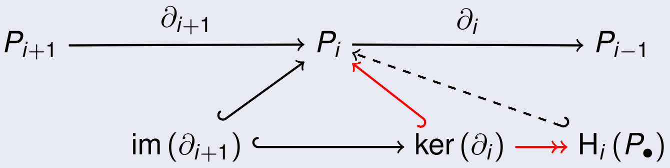

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Theorem

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Spectral sequences via generalized morphisms

Given: an excerpt of a filtered chain complex.

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & A & \xrightarrow{\partial} & B & \xrightarrow{\partial} & C & \longrightarrow & \cdots \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & A_{i+1} & \longrightarrow & B_{i+1} & \longrightarrow & C_{i+1} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & A_i & \longrightarrow & B_i & \longrightarrow & C_i & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & A_{i-1} & \longrightarrow & B_{i-1} & \longrightarrow & C_{i-1} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

Spectral sequences via generalized morphisms

We pass to its graded parts.

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & A & \xrightarrow{\partial} & B & \xrightarrow{\partial} & C & \longrightarrow & \cdots \\
 & & \vdots & & \vdots & & \vdots & & \\
\cdots & \longrightarrow & \frac{A_{i+1}}{A_i} & \xrightarrow{\overline{\partial}} & \frac{B_{i+1}}{B_i} & \xrightarrow{\overline{\partial}} & \frac{C_{i+1}}{C_i} & \longrightarrow & \cdots \\
\cdots & \longrightarrow & \frac{A_i}{A_{i-1}} & \longrightarrow & \frac{B_i}{B_{i-1}} & \longrightarrow & \frac{C_i}{C_{i-1}} & \longrightarrow & \cdots \\
\cdots & \longrightarrow & \frac{A_{i-1}}{A_{i-2}} & \longrightarrow & \frac{B_{i-1}}{B_{i-2}} & \longrightarrow & \frac{C_{i-1}}{C_{i-2}} & \longrightarrow & \cdots \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

Spectral sequences via generalized morphisms

We can compute the differentials via generalized morphisms.

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & A & \xrightarrow{\partial} & B & \xrightarrow{\partial} & C & \longrightarrow & \cdots \\
 & & \vdots & & \vdots & & \vdots & & \\
\cdots & \longrightarrow & \frac{A_{i+1}}{A_i} & \xrightarrow{\overline{\partial}} & \frac{B_{i+1}}{B_i} & \xrightarrow{\overline{\partial}} & \frac{C_{i+1}}{C_i} & \longrightarrow & \cdots \\
\cdots & \longrightarrow & \frac{A_i}{A_{i-1}} & \longrightarrow & \frac{B_i}{B_{i-1}} & \longrightarrow & \frac{C_i}{C_{i-1}} & \longrightarrow & \cdots \\
\cdots & \longrightarrow & \frac{A_{i-1}}{A_{i-2}} & \longrightarrow & \frac{B_{i-1}}{B_{i-2}} & \longrightarrow & \frac{C_{i-1}}{C_{i-2}} & \longrightarrow & \cdots \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

$\overline{\partial}$:

Spectral sequences via generalized morphisms

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$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & A & \xrightarrow{\partial} & B & \xrightarrow{\partial} & C & \longrightarrow & \cdots \\
 & & \vdots & & \vdots & & \vdots & & \\
\cdots & \longrightarrow & \frac{A_{i+1}}{A_i} & \xrightarrow{\overline{\partial}} & \frac{B_{i+1}}{B_i} & \xrightarrow{\overline{\partial}} & \frac{C_{i+1}}{C_i} & \longrightarrow & \cdots \\
\cdots & \longrightarrow & \frac{A_i}{A_{i-1}} & \longrightarrow & \frac{B_i}{B_{i-1}} & \longrightarrow & \frac{C_i}{C_{i-1}} & \longrightarrow & \cdots \\
\cdots & \longrightarrow & \frac{A_{i-1}}{A_{i-2}} & \longrightarrow & \frac{B_{i-1}}{B_{i-2}} & \longrightarrow & \frac{C_{i-1}}{C_{i-2}} & \longrightarrow & \cdots \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

$$\overline{\partial} : \frac{A_{i+1}}{A_i} \qquad\qquad \frac{B_{i+1}}{B_i}$$

Spectral sequences via generalized morphisms

We can compute the differentials via generalized morphisms.

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & A & \xrightarrow{\partial} & B & \xrightarrow{\partial} & C & \longrightarrow & \cdots \\
 & & \vdots & & \vdots & & \vdots & & \\
\cdots & \longrightarrow & \frac{A_{i+1}}{A_i} & \xrightarrow{\overline{\partial}} & \frac{B_{i+1}}{B_i} & \xrightarrow{\overline{\partial}} & \frac{C_{i+1}}{C_i} & \longrightarrow & \cdots \\
\cdots & \longrightarrow & \frac{A_i}{A_{i-1}} & \longrightarrow & \frac{B_i}{B_{i-1}} & \longrightarrow & \frac{C_i}{C_{i-1}} & \longrightarrow & \cdots \\
\cdots & \longrightarrow & \frac{A_{i-1}}{A_{i-2}} & \longrightarrow & \frac{B_{i-1}}{B_{i-2}} & \longrightarrow & \frac{C_{i-1}}{C_{i-2}} & \longrightarrow & \cdots \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

$$\overline{\partial} \; : \; \frac{A_{i+1}}{A_i} \twoheadleftarrow A_{i+1} \qquad\qquad \frac{B_{i+1}}{B_i}$$

Spectral sequences via generalized morphisms

We can compute the differentials via generalized morphisms.

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & A & \xrightarrow{\partial} & B & \xrightarrow{\partial} & C & \longrightarrow & \cdots \\
 & & \vdots & & \vdots & & \vdots & & \\
\cdots & \longrightarrow & \frac{A_{i+1}}{A_i} & \xrightarrow{\overline{\partial}} & \frac{B_{i+1}}{B_i} & \xrightarrow{\overline{\partial}} & \frac{C_{i+1}}{C_i} & \longrightarrow & \cdots \\
\cdots & \longrightarrow & \frac{A_i}{A_{i-1}} & \longrightarrow & \frac{B_i}{B_{i-1}} & \longrightarrow & \frac{C_i}{C_{i-1}} & \longrightarrow & \cdots \\
\cdots & \longrightarrow & \frac{A_{i-1}}{A_{i-2}} & \longrightarrow & \frac{B_{i-1}}{B_{i-2}} & \longrightarrow & \frac{C_{i-1}}{C_{i-2}} & \longrightarrow & \cdots \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

$$\overline{\partial} : \frac{A_{i+1}}{A_i} \twoheadleftarrow A_{i+1} \hookrightarrow A \qquad \qquad \frac{B_{i+1}}{B_i}$$

Spectral sequences via generalized morphisms

This is a generalized **subquotient embedding**.

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & A & \xrightarrow{\partial} & B & \xrightarrow{\partial} & C & \longrightarrow & \cdots \\
 & & \vdots & & \vdots & & \vdots & & \\
\cdots & \longrightarrow & \frac{A_{i+1}}{A_i} & \xrightarrow{\overline{\partial}} & \frac{B_{i+1}}{B_i} & \xrightarrow{\overline{\partial}} & \frac{C_{i+1}}{C_i} & \longrightarrow & \cdots \\
\cdots & \longrightarrow & \frac{A_{i}}{A_{i-1}} & \longrightarrow & \frac{B_{i}}{B_{i-1}} & \longrightarrow & \frac{C_{i}}{C_{i-1}} & \longrightarrow & \cdots \\
\cdots & \longrightarrow & \frac{A_{i-1}}{A_{i-2}} & \longrightarrow & \frac{B_{i-1}}{B_{i-2}} & \longrightarrow & \frac{C_{i-1}}{C_{i-2}} & \longrightarrow & \cdots \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

$$
\overline{\partial} \;:\; \frac{A_{i+1}}{A_i} \twoheadleftarrow A_{i+1} \hookrightarrow A \qquad\qquad \frac{B_{i+1}}{B_i}
$$

Spectral sequences via generalized morphisms

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & A & \xrightarrow{\partial} & B & \xrightarrow{\partial} & C & \longrightarrow & \cdots \\
 & & \vdots & & \vdots & & \vdots & & \\
\cdots & \longrightarrow & \frac{A_{i+1}}{A_i} & \xrightarrow{\overline{\partial}} & \frac{B_{i+1}}{B_i} & \xrightarrow{\overline{\partial}} & \frac{C_{i+1}}{C_i} & \longrightarrow & \cdots \\
\cdots & \longrightarrow & \frac{A_i}{A_{i-1}} & \longrightarrow & \frac{B_i}{B_{i-1}} & \longrightarrow & \frac{C_i}{C_{i-1}} & \longrightarrow & \cdots \\
\cdots & \longrightarrow & \frac{A_{i-1}}{A_{i-2}} & \longrightarrow & \frac{B_{i-1}}{B_{i-2}} & \longrightarrow & \frac{C_{i-1}}{C_{i-2}} & \longrightarrow & \cdots \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

$$\overline{\partial} : \frac{A_{i+1}}{A_i} \twoheadleftarrow A_{i+1} \hookrightarrow A \xrightarrow{\partial} B \qquad \frac{B_{i+1}}{B_i}$$

Spectral sequences via generalized morphisms

$$\cdots \longrightarrow A \xrightarrow{\partial} B \xrightarrow{\partial} C \longrightarrow \cdots$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\cdots \longrightarrow \frac{A_{i+1}}{A_i} \xrightarrow{\overline{\partial}} \frac{B_{i+1}}{B_i} \xrightarrow{\overline{\partial}} \frac{C_{i+1}}{C_i} \longrightarrow \cdots$$

$$\cdots \longrightarrow \frac{A_{i}}{A_{i-1}} \longrightarrow \frac{B_{i}}{B_{i-1}} \longrightarrow \frac{C_{i}}{C_{i-1}} \longrightarrow \cdots$$

$$\cdots \longrightarrow \frac{A_{i-1}}{A_{i-2}} \longrightarrow \frac{B_{i-1}}{B_{i-2}} \longrightarrow \frac{C_{i-1}}{C_{i-2}} \longrightarrow \cdots$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\overline{\partial} : \frac{A_{i+1}}{A_i} \twoheadleftarrow A_{i+1} \hookrightarrow A \xrightarrow{\partial} B \hookleftarrow B_{i+1} \qquad \frac{B_{i+1}}{B_i}$$

Spectral sequences via generalized morphisms

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & A & \xrightarrow{\partial} & B & \xrightarrow{\partial} & C & \longrightarrow & \cdots \\
 & & \vdots & & \vdots & & \vdots & & \\
\cdots & \longrightarrow & \frac{A_{i+1}}{A_i} & \xrightarrow{\overline{\partial}} & \frac{B_{i+1}}{B_i} & \xrightarrow{\overline{\partial}} & \frac{C_{i+1}}{C_i} & \longrightarrow & \cdots \\
\cdots & \longrightarrow & \frac{A_{i}}{A_{i-1}} & \longrightarrow & \frac{B_{i}}{B_{i-1}} & \longrightarrow & \frac{C_{i}}{C_{i-1}} & \longrightarrow & \cdots \\
\cdots & \longrightarrow & \frac{A_{i-1}}{A_{i-2}} & \longrightarrow & \frac{B_{i-1}}{B_{i-2}} & \longrightarrow & \frac{C_{i-1}}{C_{i-2}} & \longrightarrow & \cdots \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

$$
\overline{\partial} \; : \; \frac{A_{i+1}}{A_i} \twoheadleftarrow A_{i+1} \hookrightarrow A \xrightarrow{\partial} B \hookleftarrow B_{i+1} \twoheadrightarrow \frac{B_{i+1}}{B_i}
$$

Spectral sequences via generalized morphisms

This is a generalized **subquotient projection.**

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & A & \xrightarrow{\partial} & B & \xrightarrow{\partial} & C & \longrightarrow & \cdots \\
 & & \vdots & & \vdots & & \vdots & & \\
\cdots & \longrightarrow & \frac{A_{i+1}}{A_i} & \xrightarrow{\overline{\partial}} & \frac{B_{i+1}}{B_i} & \xrightarrow{\overline{\partial}} & \frac{C_{i+1}}{C_i} & \longrightarrow & \cdots \\
\cdots & \longrightarrow & \frac{A_{i}}{A_{i-1}} & \longrightarrow & \frac{B_{i}}{B_{i-1}} & \longrightarrow & \frac{C_{i}}{C_{i-1}} & \longrightarrow & \cdots \\
\cdots & \longrightarrow & \frac{A_{i-1}}{A_{i-2}} & \longrightarrow & \frac{B_{i-1}}{B_{i-2}} & \longrightarrow & \frac{C_{i-1}}{C_{i-2}} & \longrightarrow & \cdots \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

$$
\overline{\partial} : \frac{A_{i+1}}{A_i} \twoheadleftarrow A_{i+1} \hookrightarrow A \xrightarrow{\partial} B \hookleftarrow B_{i+1} \twoheadrightarrow \frac{B_{i+1}}{B_i}
$$

Spectral sequences via generalized morphisms

We can compose the arrows.

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & A & \xrightarrow{\partial} & B & \xrightarrow{\partial} & C & \longrightarrow & \cdots \\
 & & \vdots & & \vdots & & \vdots & & \\
\cdots & \longrightarrow & \frac{A_{i+1}}{A_i} & \xrightarrow{\overline{\partial}} & \frac{B_{i+1}}{B_i} & \xrightarrow{\overline{\partial}} & \frac{C_{i+1}}{C_i} & \longrightarrow & \cdots \\
\cdots & \longrightarrow & \frac{A_i}{A_{i-1}} & \longrightarrow & \frac{B_i}{B_{i-1}} & \longrightarrow & \frac{C_i}{C_{i-1}} & \longrightarrow & \cdots \\
\cdots & \longrightarrow & \frac{A_{i-1}}{A_{i-2}} & \longrightarrow & \frac{B_{i-1}}{B_{i-2}} & \longrightarrow & \frac{C_{i-1}}{C_{i-2}} & \longrightarrow & \cdots \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

$$\overline{\partial} : \frac{A_{i+1}}{A_i} \twoheadleftarrow A_{i+1} \hookrightarrow A \xrightarrow{\partial} B \hookleftarrow B_{i+1} \twoheadrightarrow \frac{B_{i+1}}{B_i}$$

Spectral sequences via generalized morphisms

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$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & A & \xrightarrow{\partial} & B & \xrightarrow{\partial} & C & \longrightarrow & \cdots \\
 & & \vdots & & \vdots & & \vdots & & \\
\cdots & \longrightarrow & \frac{A_{i+1}}{A_i} & \xrightarrow{\overline{\partial}} & \frac{B_{i+1}}{B_i} & \xrightarrow{\overline{\partial}} & \frac{C_{i+1}}{C_i} & \longrightarrow & \cdots \\
\cdots & \longrightarrow & \frac{A_i}{A_{i-1}} & \longrightarrow & \frac{B_i}{B_{i-1}} & \longrightarrow & \frac{C_i}{C_{i-1}} & \longrightarrow & \cdots \\
\cdots & \longrightarrow & \frac{A_{i-1}}{A_{i-2}} & \longrightarrow & \frac{B_{i-1}}{B_{i-2}} & \longrightarrow & \frac{C_{i-1}}{C_{i-2}} & \longrightarrow & \cdots \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

$$
\overline{\partial} : \frac{A_{i+1}}{A_i} \twoheadleftarrow A_{i+1} \hookrightarrow A \xrightarrow{\partial} B \hookleftarrow B_{i+1} \twoheadrightarrow \frac{B_{i+1}}{B_i}
$$

Spectral sequences via generalized morphisms

We can compose the arrows.

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & A & \xrightarrow{\partial} & B & \xrightarrow{\partial} & C & \longrightarrow & \cdots \\
 & & \vdots & & \vdots & & \vdots & & \\
\cdots & \longrightarrow & \frac{A_{i+1}}{A_i} & \xrightarrow{\overline{\partial}} & \frac{B_{i+1}}{B_i} & \xrightarrow{\overline{\partial}} & \frac{C_{i+1}}{C_i} & \longrightarrow & \cdots \\
\cdots & \longrightarrow & \frac{A_i}{A_{i-1}} & \longrightarrow & \frac{B_i}{B_{i-1}} & \longrightarrow & \frac{C_i}{C_{i-1}} & \longrightarrow & \cdots \\
\cdots & \longrightarrow & \frac{A_{i-1}}{A_{i-2}} & \longrightarrow & \frac{B_{i-1}}{B_{i-2}} & \longrightarrow & \frac{C_{i-1}}{C_{i-2}} & \longrightarrow & \cdots \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

$$
\overline{\partial} \;:\; \frac{A_{i+1}}{A_i} \twoheadleftarrow A_{i+1} \hookrightarrow A \xrightarrow{\partial} B \hookleftarrow B_{i+1} \twoheadrightarrow \frac{B_{i+1}}{B_i}
$$

Spectral sequences via generalized morphisms

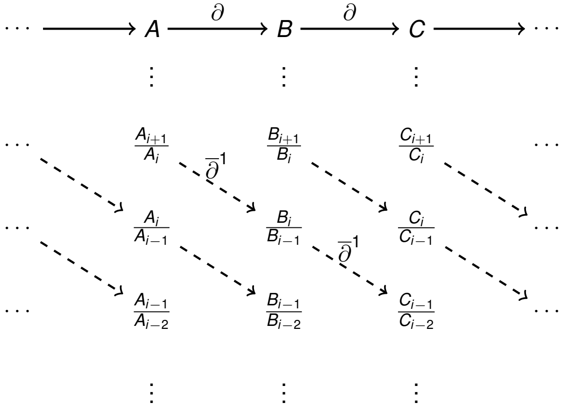
This formula still makes sense if we map 1 step down.

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & A & \xrightarrow{\partial} & B & \xrightarrow{\partial} & C & \longrightarrow & \cdots \\
 & & \vdots & & \vdots & & \vdots & & \\
\cdots & \longrightarrow & \frac{A_{i+1}}{A_i} & \xrightarrow{\overline{\partial}} & \frac{B_{i+1}}{B_i} & \xrightarrow{\overline{\partial}} & \frac{C_{i+1}}{C_i} & \longrightarrow & \cdots \\
\cdots & \longrightarrow & \frac{A_i}{A_{i-1}} & \longrightarrow & \frac{B_i}{B_{i-1}} & \longrightarrow & \frac{C_i}{C_{i-1}} & \longrightarrow & \cdots \\
\cdots & \longrightarrow & \frac{A_{i-1}}{A_{i-2}} & \longrightarrow & \frac{B_{i-1}}{B_{i-2}} & \longrightarrow & \frac{C_{i-1}}{C_{i-2}} & \longrightarrow & \cdots \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

$$\overline{\partial} : \frac{A_{i+1}}{A_i} \twoheadleftarrow A_{i+1} \hookrightarrow A \xrightarrow{\partial} B \hookleftarrow B_{i+1} \twoheadrightarrow \frac{B_{i+1}}{B_i}$$

Spectral sequences via generalized morphisms

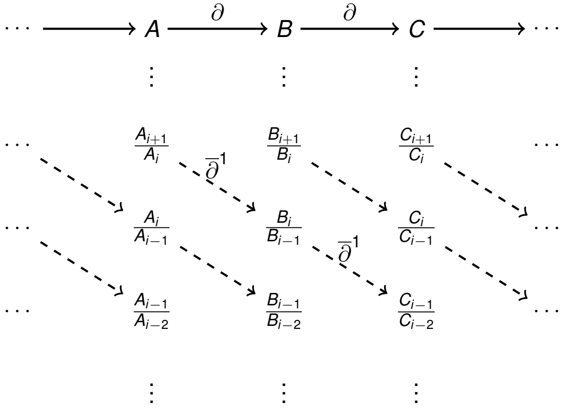
This formula still makes sense if we map 1 step down.



$$\overline{\partial}^1 : \frac{A_{i+1}}{A_i} \twoheadleftarrow A_{i+1} \hookrightarrow A \xrightarrow{\partial} B \hookleftarrow B_i \twoheadrightarrow \frac{B_i}{B_{i-1}}$$

Spectral sequences via generalized morphisms

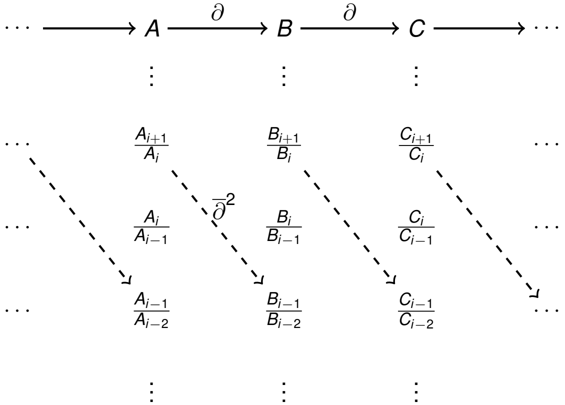
One more step ...



$$\overline{\partial}^1 : \frac{A_{i+1}}{A_i} \twoheadleftarrow A_{i+1} \hookrightarrow A \xrightarrow{\partial} B \hookleftarrow B_i \twoheadrightarrow \frac{B_i}{B_{i-1}}$$

Spectral sequences via generalized morphisms

One more step ...



$$\overline{\partial}^2 : \frac{A_{i+1}}{A_i} \twoheadleftarrow A_{i+1} \hookrightarrow A \xrightarrow{\partial} B \hookleftarrow B_{i-1} \twoheadrightarrow \frac{B_{i-1}}{B_{i-2}}$$

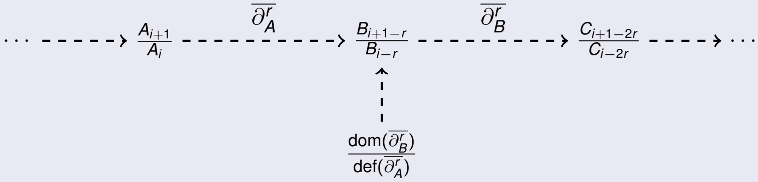
Spectral sequences via generalized morphisms

For all $r \geq 0$, we get so-called generalized chain complexes.

$$\cdots \dashrightarrow \frac{A_{i+1}}{A_i} \overset{\overline{\partial^r_A}}{\dashrightarrow} \frac{B_{i+1-r}}{B_{i-r}} \overset{\overline{\partial^r_B}}{\dashrightarrow} \frac{C_{i+1-2r}}{C_{i-2r}} \dashrightarrow \cdots$$

Spectral sequences via generalized morphisms

For all $r \geq 0$, we get so-called generalized chain complexes.



Spectral sequences via generalized morphisms

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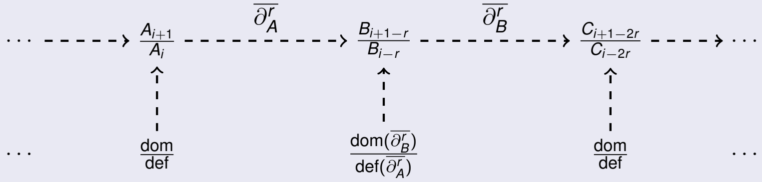


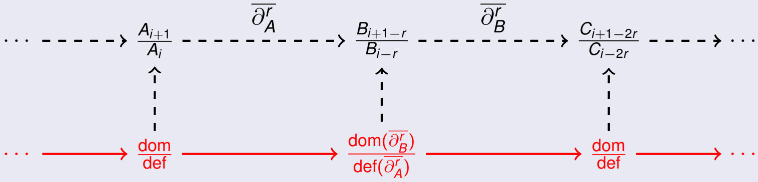
Spectral sequences via generalized morphisms

For all $r \geq 0$, we get so-called generalized chain complexes.

$$
\begin{array}{ccccccccc}
\cdots & \dashrightarrow & \frac{A_{i+1}}{A_i} & \overset{\overline{\partial^r_A}}{\dashrightarrow} & \frac{B_{i+1-r}}{B_{i-r}} & \overset{\overline{\partial^r_B}}{\dashrightarrow} & \frac{C_{i+1-2r}}{C_{i-2r}} & \dashrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & \frac{\mathrm{dom}}{\mathrm{def}} & \longrightarrow & \frac{\mathrm{dom}(\overline{\partial^r_B})}{\mathrm{def}(\overline{\partial^r_A})} & \longrightarrow & \frac{\mathrm{dom}}{\mathrm{def}} & \longrightarrow & \cdots
\end{array}
$$

Spectral sequences via generalized morphisms

For all $r \geq 0$, we get so-called generalized chain complexes.



- These are the chain complexes on the r-th page of the associated **spectral sequence.**

Spectral sequences via generalized morphisms

For all $r \geq 0$, we get so-called generalized chain complexes.

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & C_{j+1} & \longrightarrow & C_j & \longrightarrow & C_{j-1} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \dashrightarrow & \frac{F_{i+1}C_{j+1}}{F_iC_{j+1}} & \dashrightarrow & \frac{F_{i+1-r}C_j}{F_{i-r}C_j} & \dashrightarrow & \frac{F_{i+1-2r}C_{j-1}}{F_{i-2r}C_{j-1}} & \dashrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & E^r_{i+1,j-i} & \longrightarrow & E^r_{i+1-r,j-i+(r-1)} & \longrightarrow & E^r_{i+1-2r,j-i+2(r-1)} & \rightarrow & \cdots
\end{array}
$$

- These are the chain complexes on the r-th page of the associated **spectral sequence.**

Spectral sequences via generalized morphisms

For all $r \geq 0$, we get so-called generalized chain complexes.

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & C_{j+1} & \longrightarrow & C_j & \longrightarrow & C_{j-1} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \dashrightarrow & \frac{F_{i+1}C_{j+1}}{F_iC_{j+1}} & \dashrightarrow & \frac{F_{i+1-r}C_j}{F_{i-r}C_j} & \dashrightarrow & \frac{F_{i+1-2r}C_{j-1}}{F_{i-2r}C_{j-1}} & \dashrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & E^r_{i+1,j-i} & \longrightarrow & E^r_{i+1-r,j-i+(r-1)} & \longrightarrow & E^r_{i+1-2r,j-i+2(r-1)} & \to & \cdots
\end{array}$$

- These are the chain complexes on the r-th page of the associated **spectral sequence.**
- We just computed them **without a recursive strategy**.

Spectral sequences

Convergence

Let $C_\bullet := 0 = F_{-n-1}C_\bullet \leq F_{-n}C_\bullet \leq \cdots \leq F_0 C_\bullet$ be a finitely filtered complex

Spectral sequences

Convergence

Let $C_\bullet := 0 = F_{-n-1}C_\bullet \leq F_{-n}C_\bullet \leq \cdots \leq F_0C_\bullet$ be a finitely filtered complex and for $p, q \in \mathbb{Z}$, $r \geq 0$ let E^r_{pq} be the **computed** objects

Spectral sequences

Convergence

Let $C_\bullet := 0 = F_{-n-1}C_\bullet \leq F_{-n}C_\bullet \leq \cdots \leq F_0 C_\bullet$ be a finitely filtered complex and for $p, q \in \mathbb{Z}$, $r \geq 0$ let E^r_{pq} be the **computed** objects with their generalized embeddings

$$E^r_{pq} \hookrightarrow C_{p+q}.$$

Spectral sequences

Convergence

Let $C_\bullet := 0 = F_{-n-1}C_\bullet \leq F_{-n}C_\bullet \leq \cdots \leq F_0 C_\bullet$ be a finitely filtered complex and for $p, q \in \mathbb{Z}$, $r \geq 0$ let E^r_{pq} be the **computed** objects with their generalized embeddings

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Then for all p, q and all $k \geq n+1$ we have

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Then for all p, q and all $k \geq n+1$ we have

$$E^k_{pq} \cong E^{n+1}_{pq}$$

Spectral sequences

Convergence

Let $C_\bullet := 0 = F_{-n-1}C_\bullet \leq F_{-n}C_\bullet \leq \cdots \leq F_0 C_\bullet$ be a finitely filtered complex and for $p, q \in \mathbb{Z}$, $r \geq 0$ let E^r_{pq} be the **computed** objects with their generalized embeddings

$$E^r_{pq} \hookrightarrow C_{p+q}.$$

Then for all p, q and all $k \geq n+1$ we have

$$E^k_{pq} \cong E^{n+1}_{pq} =: E^\infty_{pq}.$$

Filtration morphisms

Using the above generalized embedding and the generalized projection to the homology,

Filtration morphisms

Using the above generalized embedding and the generalized projection to the homology, we get a generalized morphism

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$$E^{\infty}_{p,q} \dashrightarrow \mathrm{H}_{p+q}(C_\bullet)$$

Filtration morphisms

Using the above generalized embedding and the generalized projection to the homology, we get a generalized morphism

$$E^{\infty}_{p,q} \hookrightarrow C_{p+q} \qquad \mathrm{H}_{p+q}(C_{\bullet})$$

$$E^{\infty}_{p,q} \dashrightarrow \mathrm{H}_{p+q}(C_{\bullet})$$

Filtration morphisms

Using the above generalized embedding and the generalized projection to the homology, we get a generalized morphism

$$E^{\infty}_{p,q} \hookrightarrow C_{p+q} \twoheadrightarrow \mathrm{H}_{p+q}(C_\bullet)$$

with a dashed arrow $E^{\infty}_{p,q} \dashrightarrow \mathrm{H}_{p+q}(C_\bullet)$.

Filtration morphisms

Using the above generalized embedding and the generalized projection to the homology, we get a generalized morphism

$$E^{\infty}_{p,q} \dashrightarrow C_{p+q} \dashrightarrow \mathrm{H}_{p+q}(C_{\bullet})$$

Filtration morphisms

Using the above generalized embedding and the generalized projection to the homology, we get a generalized morphism

$$E^{\infty}_{p,q} \xleftarrow{\alpha} X \xrightarrow{\beta} \mathrm{H}_{p+q}(C_{\bullet})$$

Filtration morphisms

Using the above generalized embedding and the generalized projection to the homology, we get a generalized morphism

$$E^{\infty}_{p,q} \xleftarrow{\ \alpha\ } X \xrightarrow{\ \beta\ } \mathrm{H}_{p+q}\left(C_\bullet\right)$$

This induces a filtration on $H := \mathrm{H}_{p+q}\left(C_\bullet\right)$ with

Filtration morphisms

Using the above generalized embedding and the generalized projection to the homology, we get a generalized morphism

$$E^{\infty}_{p,q} \xleftarrow{\ \alpha\ } X \xrightarrow{\ \beta\ } \mathrm{H}_{p+q}\left(C_\bullet\right)$$

This induces a filtration on $H := \mathrm{H}_{p+q}\left(C_\bullet\right)$ with

$$\begin{aligned} F_pH/F_{p-1}H &\cong E^{\infty}_{p,q} \\ F_pH &\cong \mathrm{im}\left(\beta\right). \end{aligned}$$

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(with a dashed arrow $E^{\infty}_{-p,p} \dashrightarrow M$; the arrows $E^{\infty}_{-p,p} \hookrightarrow C_0$ and $C_0 \twoheadrightarrow \mathrm{H}_0(C_\bullet)$ are dashed)

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$$F_{-p}M / F_{-p-1}M \cong E^{\infty}_{-p,p}$$
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Presentations from filtrations

Let $F_{-n}M \leq F_{-n+1}M \leq \cdots \leq F_0 M := M$ be a finitely presented filtered module.

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If $\mathtt{M}_i$ is a presentation matrix for $F_iM/F_{i-1}M$, then M can be presented by an upper block triangular matrix

$$\begin{pmatrix} \mathtt{M}_0 & * & \dots & \dots & * \\ & \mathtt{M}_{-1} & * & \dots & * \\ & & \ddots & \ddots & \vdots \\ & & & \mathtt{M}_{-n+1} & * \\ & & & & \mathtt{M}_{-n} \end{pmatrix}.$$

Example: filtered presentation

Consider the module with relations

$$\begin{pmatrix} 0 & 0 & 0 & 0 & xz & -z^2 \\ 0 & 0 & 0 & 0 & xy & -yz \\ 0 & -x^2z+xyz+xz^2 & y^2z & -xz+yz & x-y & 0 \\ 0 & 0 & 0 & 0 & x^2 & -xz \\ -xy & -x^3+x^2y+x^2z & xy^2 & -x^2+xy & 0 & x-y \\ z & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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Computing the purity filtration by using the bidualizing spectral sequence yields

$$\left(\begin{array}{ccccc|c|c} x & -z & 0 & 0 & 0 & 0 & 1 \\ -y & z & y^2z & -yz^2 & -xz+yz & 0 & -1 \\ 0 & x-y & xy^2 & -xyz & -x^2+xy & xy & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & z & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 & 0 & 0 & y \\ 0 & 0 & 0 & 0 & 0 & 0 & x \end{array}\right)$$

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