Motives, algorithms and programming

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mathematics: following F. Brown (Oxford), C. Dupont (Montpellier)

programming: joint with C. Dupont (Montpellier), M. Barakat (Siegen) and kind & efficient support from the CAP team!

[1] Fusion of Université Paris 4 and Université Paris 6 Pierre et Marie Curie

[2] To be merged next year with Université Paris 5 René Descartes, into Université de Paris!

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3. $\dim_{\mathbb{Q}} \langle \zeta(3), \zeta(5), \zeta(7), \dots \rangle_{\mathbb{Q}} = \infty$ (Ball-Rivoal 2000)
4. at least one of $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, $\zeta(11)$ is irrational (Zudilin 2004)

Multiple zeta values

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- Many relations, but graded dimension is predictable.
- Linear (in)dependence is easier than algebraic independence.

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Contradiction if r and ε are sufficiently small, so that $e^r\varepsilon < 1$.

Irrationality of $\zeta(3)$

Beuker's integral:

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&= a_n \zeta(3) + b_n
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with $a_n \in \mathbb{Z}$ and $D_n^3 b_n \in \mathbb{Z}$, bounded by

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hence $\zeta(3)$ is irrational!

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\mathcal{M}_{0,N} &= \{\text{curves of genus 0 with } N \text{ ordered marked points}\} \\
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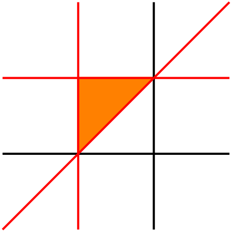
Example: $N = 5$, $n = 2$



A recipe: periods of moduli spaces $\mathcal{M}_{0,N}$

Examples of period integrals on $\mathcal{M}_{0,N}$:

$$\int_{\delta_n} \prod_i t_i^{a_i} \prod_j (1 - t_j)^{b_j} \prod_{i<j} (t_i - t_j)^{c_{i,j}} \, \mathrm{d}t_1 \dots \mathrm{d}t_n$$

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General recipe for linear forms in MZV's

Consider family of convergent integrals

$$I_{f,\omega}(k) = \int_{\delta_n} f^k \omega$$

where $\omega \in \Omega^n(\mathcal{M}_{0,N}, \mathbb{Q})$ is a regular n-form and $f \in \Omega^0(\mathcal{M}_{0,N}, \mathbb{Q})$.

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In terms of algebraic geometry: consider the (mixed Tate) motive

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Then $\mathrm{gr}_{2k}^W \mathrm{H}_{A,B} = 0 \Longrightarrow$ vanishing of coefficients $a_j^{(i)}$ in weight k.

Periods and cohomology

For a smooth algebraic variety defined over $\mathbb{Q}$, we have:

- the Betti cohomology groups (singular cohomology) $\mathsf{H}^k_{\mathrm{B}}(X)$;
- the algebraic de Rham cohomology groups $\mathsf{H}^k_{\mathrm{dR}}(X)$;
- the comparison isomorphism $\mathsf{H}^k_{\mathrm{B}}(X) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} \mathsf{H}^k_{\mathrm{dR}}(X) \otimes_{\mathbb{Q}} \mathbb{C}$, whose coefficients are *periods*. Equivalently: Betti / de Rham *pairing*.

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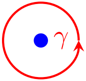

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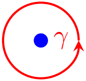
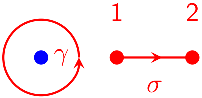


$$\begin{array}{cc} & dz/z \\ \gamma & (2\pi i) \end{array}$$

$K_2 = \mathrm{H}^1(\mathbb{C}^*, \{1, 2\})$



$$\begin{array}{cc} & \begin{array}{cc} dz & dz/z \end{array} \\ \begin{array}{c} \sigma \\ \gamma \end{array} & \begin{pmatrix} 1 & \log 2 \\ 0 & 2\pi i \end{pmatrix} \end{array}$$

$0 \to \mathbb{Q}(0) \to K_2 \to \mathbb{Q}(-1) \to 0,$ "ramified at 2"

Mixed Tate motives over $\mathbb{Q}$, over $\mathbb{Z}$ ("non-ramified")

Category *MTM*($\mathbb{Q}$): abelian, rigid tensor category (symmetric, duals), exact faithful tensor functors $\omega_{\mathrm{dR}}, \omega_{\mathrm{B}}$: *MTM*($\mathbb{Q}$) $\to \mathrm{Vect}$ (Tannakian)

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- $\mathrm{Per}(MTM(\mathbb{Z})) = \bigcup_N \mathrm{Per}(\mathcal{M}_{0,N}) = \mathbb{Q}[2\pi i][\mathrm{MZV}] = \mathbb{Q}[2\pi i][\mathrm{MZV}_{2,3}]$

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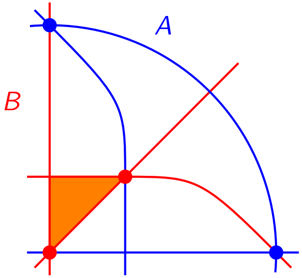

$\mathbb{P}^2$

6 lines, 7 points

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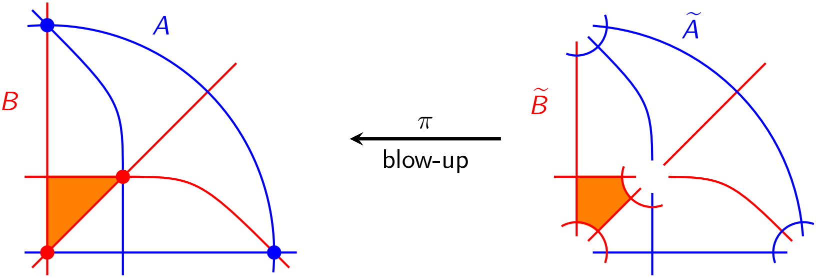

$$\mathbb{P}^2 \xleftarrow{\ \pi\ } \widetilde{\mathbb{P}^2} = \overline{\mathcal{M}}_{0,5}$$

6 lines, 7 points — 10 lines, 15 points

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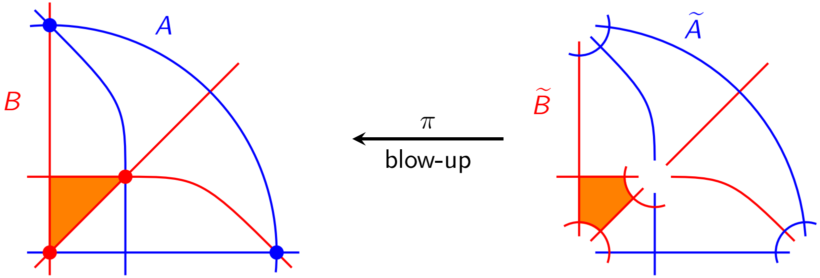

$\mathbb{P}^2$ $\xleftarrow{\pi}$ $\widetilde{\mathbb{P}^2} = \overline{\mathcal{M}}_{0,5}$

6 lines, 7 points

10 lines, 15 points

$\mathrm{H} := \mathrm{H}^2(\widetilde{\mathbb{P}^2} \setminus \widetilde{A}, \widetilde{B} \setminus \widetilde{A})$. Period matrix: $\begin{pmatrix} 1 & \zeta(2) \\ 0 & (2\pi i)^2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & (2\pi i)^2 \end{pmatrix}$.

Bi-arrangements of hyperplanes

Definition (Dupont 2014)

A *bi-arrangement of hyperplanes* is a triple $(\mathcal{L}, \mathcal{M}, \chi)$ where

- $\mathcal{L} = \{L_1, \ldots, L_l\}$ is a set of hyperplanes in $\mathbb{P}^n$;
- $\mathcal{M} = \{M_1, \ldots, M_m\}$ is a set of hyperplanes in $\mathbb{P}^n$;
- $\chi : \mathcal{S} = \mathrm{Flats}(A \cup B) \to \{\lambda, \mu\}$ is a coloring function, satisfying

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Inspired by (Aomoto 1977, 1982) and (Beilinson-Goncharov-Schechtman-Varchenko, 1989).

The Orlik-Solomon bicomplex

Definition

We define the *Orlik-Solomon bicomplex* $A_{\bullet,\bullet} = A_{\bullet,\bullet}(\mathcal{L}, \mathcal{M}, \chi)$:

$$\begin{tikzcd}
{} \arrow[r, dotted] & A_{2,0} \arrow[r] \arrow[d, dotted] & A_{1,0} \arrow[r, "d'"] \arrow[d] & A_{0,0} \arrow[d, "d''"] \\
& {} \arrow[r, dotted] & A_{1,1} \arrow[r] \arrow[d, dotted] & A_{0,1} \arrow[d] \\
& & {} \arrow[r, dotted] & A_{0,2} \arrow[d, dotted] \\
& & & {}
\end{tikzcd}$$

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 & & & {}
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$$

We define $A_{i,j} = \bigoplus_{S \in \mathcal{S}_{i+j}} A_{i,j}^S$ and the differentials d' and d'' by induction on the codimension $i + j$. Here $\mathcal{S}_k$ = flats of codimension k.

The Orlik-Solomon bicomplex

Base step of the induction : $A_{0,0} = \mathbb{Q}$.

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Inductive step :

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Hence we use:

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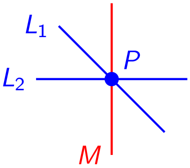
- `MorphismBetweenDirectSums`,
  `ComponentOfMorphismIntoDirectSum`,
  `ComponentOfMorphismFromDirectSum`...

Example



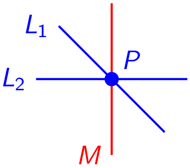
$$\begin{tikzcd}[column sep=small]
 & P \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
L_1 \arrow[dr, dash] & L_2 \arrow[d, dash] & M \arrow[dl, dash] \\
 & X &
\end{tikzcd}$$

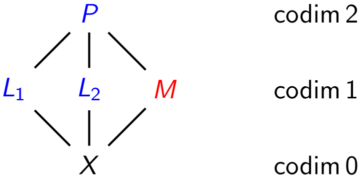
codim 2

codim 1

codim 0

Example

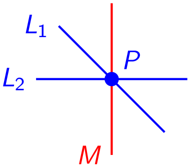




$A^{\leq X}_{\bullet,\bullet}$

$\mathbb{Q}$

$$A^{X}_{0,0} = \mathbb{Q}$$



$$
\begin{tikzcd}
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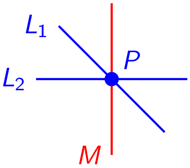
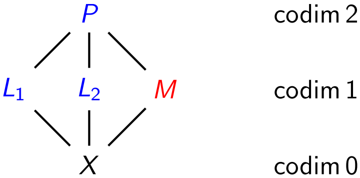
$A^{<L_i}_{\bullet,\bullet}$

$\mathbb{Q}$

$$A^X_{0,0} = \mathbb{Q}$$

Example



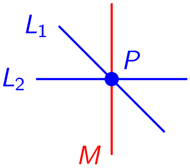


$A_{\bullet,\bullet}^{\leq L_i}$

$$\mathbb{Q} \xrightarrow{(\,1\,)} \mathbb{Q}$$

$$A_{0,0}^{X} = \mathbb{Q}$$

$$A_{1,0}^{L_i} = \mathbb{Q} \qquad d_{1,0}^{\prime X, L_i} = (1)$$



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codim 2

codim 1

codim 0

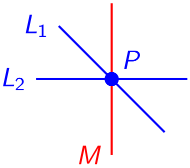
$A^{<M}_{\bullet,\bullet}$

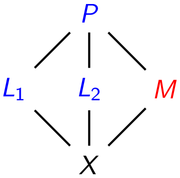
$\mathbb{Q}$

$$A^{X}_{0,0} = \mathbb{Q}$$

$$A^{L_i}_{1,0} = \mathbb{Q} \qquad d'^{X,L_i}_{1,0} = (1)$$

Example





codim 2

codim 1

codim 0

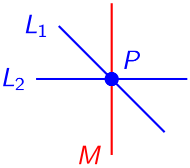
$$A^{\le M}_{\bullet,\bullet}$$

$$\begin{tikzcd} \mathbb{Q} \arrow[d, "(1)"] \\ \mathbb{Q} \end{tikzcd}$$

$$A^{X}_{0,0} = \mathbb{Q}$$

$$A^{L_i}_{1,0} = \mathbb{Q} \qquad d'^{X,L_i}_{1,0} = (1)$$

$$A^{M}_{1,0} = \mathbb{Q} \qquad d''^{M,X}_{1,0} = (1)$$



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$$A^{<P}_{\bullet,\bullet}$$

$$\begin{tikzcd} \mathbb{Q}^2 \arrow[r, "\begin{pmatrix} 1 & 1 \end{pmatrix}"] & \mathbb{Q} \arrow[d, "\begin{pmatrix} 1 \end{pmatrix}"] \\ & \mathbb{Q} \end{tikzcd}$$

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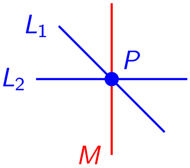
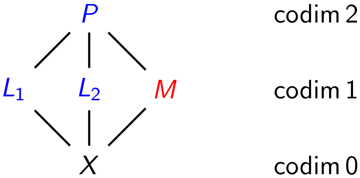
Example





$A^{\le P}_{\bullet,\bullet}$

$$
\begin{tikzcd}
\mathbb{Q} \arrow[r, "\begin{pmatrix} 1 \\ -1 \end{pmatrix}"] & \mathbb{Q}^2 \arrow[r, "\begin{pmatrix} 1 & 1 \end{pmatrix}"] & \mathbb{Q} \arrow[d, "\begin{pmatrix} 1 \end{pmatrix}"] \\
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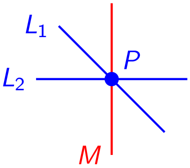

$$
\begin{tikzcd}[row sep=small]
 & P & & \text{codim } 2 \\
L_1 \arrow[ur, dash] & L_2 \arrow[u, dash] & M \arrow[ul, dash] & \text{codim } 1 \\
 & X \arrow[ul, dash] \arrow[u, dash] \arrow[ur, dash] & & \text{codim } 0
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Exactness

Definition

A bi-arrangement of hyperplanes $(\mathcal{L}, \mathcal{M}, \chi)$ is *exact* if the above exact sequences can be continued to long exact sequences

$$0 \to A^{\Sigma}_{i,j} \xrightarrow{d'} \bigoplus_{S \supset \Sigma} A^{S}_{i-1,j} \xrightarrow{d'} \bigoplus_{T \supset \Sigma} A^{T}_{i-2,j} \xrightarrow{d'} \cdots$$

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– Deletion and restriction formalism for exact bi-arrangements of hyperplanes.

The main theorem

Theorem (Dupont 2014)

For an *exact* bi-arrangement of hyperplanes $(\mathcal{L}, \mathcal{M}, \chi)$ in $\mathbb{P}^n$, "the Orlik-Solomon bicomplex $A_{\bullet,\bullet}(\mathcal{L}, \mathcal{M}, \chi)$ computes the motive $H^\bullet(\mathcal{L}, \mathcal{M}, \chi)$".

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- we consider the double complex $A_{0 \leq \bullet \leq k, 0 \leq \bullet \leq n-k}$;
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- The weight-graded quotients $\mathrm{gr}^W_{2k} H^\bullet(\mathcal{L}, \mathcal{M}, \chi)$ are combinatorial invariants, but not the whole motive $H^\bullet(\mathcal{L}, \mathcal{M}, \chi)$.

Explicit computations: the tame case

Combinatorial notion of *tame* bi-arrangements of hyperplanes.

- Generic bi-arrangements are tame
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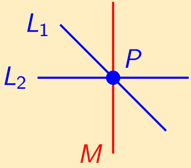

$$A_{\bullet,\bullet} = \Lambda^\bullet(e_1, e_2) \otimes \Lambda^\bullet(f_1^\vee)/(d(e_1 \wedge e_2) \otimes f_1^\vee)$$

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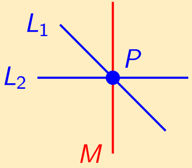

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Example

One can define multiple zeta bi-arrangements $\mathcal{Z}(n_1, \ldots, n_r)$ that are tame.

Basic cellular integrals

Given a permutation $\sigma \in \mathfrak{S}_N$, define on $\mathbb{P}^N \setminus \bigcup \{z_i = z_j\}$:

$$\tilde{f}_\sigma = \prod_{i \in \mathbb{Z}/N\mathbb{Z}} \frac{z_i - z_{i+1}}{z_{\sigma(i)} - z_{\sigma(i+1)}} \quad \text{and} \quad \tilde{\omega}_\sigma = \frac{\mathrm{d}z_1 \ldots \mathrm{d}z_N}{\prod_{i \in \mathbb{Z}/N\mathbb{Z}} \left(z_{\sigma(i)} - z_{\sigma(i+1)}\right)},$$

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$N = 5$: only ${}_5\pi = [5, 2, 4, 1, 3]$, $\quad N = 6$: only ${}_6\pi = [6, 2, 4, 1, 5, 3]$

Vanishing for basic cellular integrals

Theorem (Brown 2016)

Suppose that $A, B \subset \mathcal{M}_{0,N}$ are cellular boundary divisors with no common irreducible components. Let $n = N - 3$. Then

$$\mathrm{gr}_2^W \mathrm{H}_{A,B} = \mathrm{gr}_{2n-2}^W \mathrm{H}_{A,B} = 0$$

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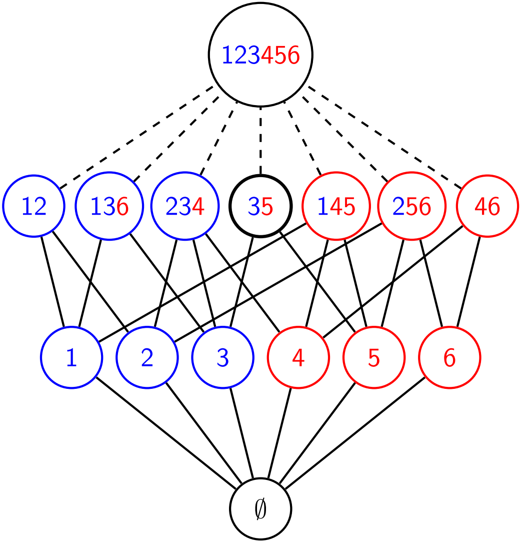
Those are the Apéry motives! They give the linear combinations of 1 and $\zeta(2)$ for $N = 5$, resp. 1 and $\zeta(3)$ for $N = 6$, used in the irrationality proofs.

Flat poset for $\zeta(2)$



35 may be set red or blue

morphism red $\to$ blue

`KernelObjectFunctorial`
`TotalComplexFunctorial`

Take the image!

Irrelevant for $\zeta(2)$:
$101 \twoheadrightarrow 101 \hookrightarrow 101$

Relevant for $\zeta(3)$:

$1011 \twoheadrightarrow 1001 \hookrightarrow 1101$

More basic cellular integrals

$N = 7$

Two dual pairs and one self-dual configuration:

$$
\begin{array}{c}
{}_7\pi_1 = [7,2,4,1,6,3,5] \quad \longleftrightarrow \quad {}_7\pi_1^\vee = [7,2,5,1,4,6,3] \\
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$N = 8$

Among the 17 convergent configurations, let us note

$${}_8\pi_8 = [8,2,5,1,6,4,7,3] \longleftrightarrow {}_8\pi_8^\vee = [8,2,4,1,7,5,3,6]$$

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