# Implementation of Quillen model categories

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# Overview

1. Preliminaries
   - Model categories
   - Homotopy categories of Model Categories

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   - Model categories
   - Homotopy categories of Model Categories
2. Complex categories
   - Definitions and constructions
   - Bounded derived category of finitely presented modules
   - Bounded derived category of acyclic quiver representations

# Overview

# Retracts

## Definition

Let $\mathbf{C}$ be a category and $X, Y$ objects in $\mathbf{C}$. We say $X$ is retract of $Y$ if there exist morphisms $i : X \to Y$ and $r : Y \to X$ such that $ir = \mathrm{id}_X$.

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I.e., there exists a commutative diagram in $\mathbf{C}$

$$\begin{array}{ccccc} X & \longrightarrow & X' & \longrightarrow & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Y & \longrightarrow & Y' & \longrightarrow & Y \end{array}$$

in which the rows compose to the identity morphisms $\mathrm{id}_X$ and $\mathrm{id}_Y$.

# Quillen Model categories

## Definition

A Model category is a category $\mathbf{C}$ with three distinguished classes of morphisms

1. weak equivalences $\mathrm{WE}$ (denoted by $\xrightarrow{\sim}$),
2. fibrations $\mathrm{Fib}$ (denoted by $\twoheadrightarrow$) and
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which are closed under composition, contain all the isomorphisms, and satisfy the following axioms:

- (*Limit axiom*) The category $\mathbf{C}$ has all finite limits and colimits. In particular, $\mathbf{C}$ has both initial and terminal objects, denoted by $e$ and $*$ respectively.

# Quillen Model categories

## Definition (continued)

- (*2-out-of-3 axiom*) If $f : X \to Y$ and $g : Y \to Z$ are two morphisms in $\mathbf{C}$, then if two out of the three morphisms $f, g, fg$ are in $\mathrm{WE}$ then so is the third.

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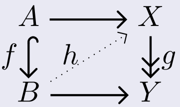
- (*Retract axiom*) The classes $\mathrm{WE}, \mathrm{Fib}$ and $\mathrm{Cof}$ are closed under taking retracts.
- (*Lifting axiom*) Given a commutative square diagram




with $f \in \mathrm{Cof}$ and $g \in \mathrm{Fib}$. If one of the morphisms $f$ or $g$ is in addition a weak equivalence, then there exists a morphism $h : B \to X$ making the resulting diagram commutative. $h$ is called a lifting of the diagram.

# Quillen Model categories

## Definition

- (*Factorization axiom*) Every morphism $f : A \to X$ in $\mathbf{C}$ can be decomposed in two ways:

$$A \xhookrightarrow{\sim} B \twoheadrightarrow X$$

$$A \hookrightarrow Y \xrightarrow{\sim} \!\!\!\!\rightarrow X$$

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- For any object $X$ in $\mathbf{C}$ we can apply the Factorization-axiom on $e \to X$ and $X \to *$ to get

$$e \hookrightarrow QX \xrightarrow[p_X]{\sim} \!\!\!\!\rightarrow X$$

$$X \xhookrightarrow[i_X]{\sim} RX \longrightarrow\!\!\!\!\rightarrow *.$$

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# Replacements of a morphism

**Lemma**

*For any morphism $f : X \to Y$ in $\mathbf{C}$, we have two morphisms $Qf : QX \to QY$ and $Rf : RX \to RY$ such that the following diagram is commutative*

$$\begin{array}{ccccc} QX & \overset{\sim}{\underset{p_X}{\twoheadrightarrow}} & X & \overset{\sim}{\underset{i_X}{\hookrightarrow}} & RX \\ {\scriptstyle Qf}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle Rf} \\ QY & \overset{\sim}{\underset{p_Y}{\twoheadrightarrow}} & Y & \overset{\sim}{\underset{i_Y}{\hookrightarrow}} & RY \end{array}$$

# Replacements of a morphism

## Lemma

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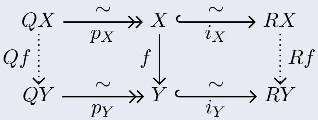
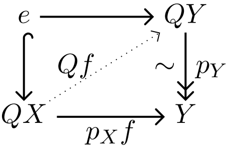



Proof. The existence follows from Lifting axiom. For $Qf$ we use the following input



# Cylinder object and Left homotopic morphisms

## Definition

Let $A$ be an object of $\mathbf{C}$. Let $(A \amalg A, j_1, j_2 : A \to A \amalg A)$ denote the pushout of $(e \to A, e \to A)$. The folding morphism of $A$ is the universal morphism from pushout $\nabla : A \amalg A \to A$ dertermined by $(\mathrm{id}_A, \mathrm{id}_A)$. A cylinder object for $A$ is an object $\mathbf{Cyl}(A)$ of $\mathbf{C}$ together with a factorization

$$A \amalg A \xrightarrow[i]{} \mathbf{Cyl}(A) \xrightarrow[p]{\sim} A$$

of the folding morphism.

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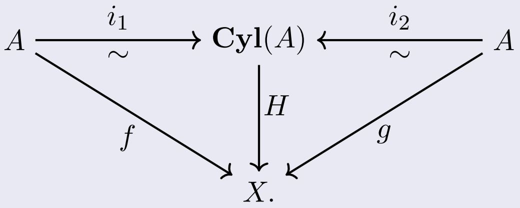
We set $i_1 := j_1 i$ and $i_2 := j_2 i$.

# Cylinder object and Left homotopic morphisms

## Definition

If $f, g \in \mathrm{Hom}_{\mathbf{C}}(A, X)$, a left homotopy from $f$ to $g$ (written $f \overset{l}{\sim} g$) is defined to be a morphism $H : \mathbf{Cyl}(A) \to X$ such that the following diagram commutes.



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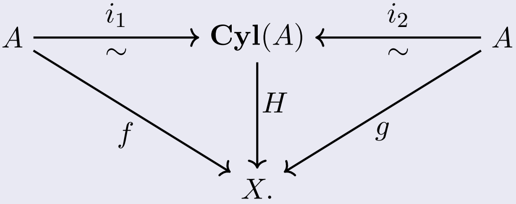


A direct consequence is

$$f \in \mathrm{WE} \iff H \in \mathrm{WE} \iff g \in \mathrm{WE}\,.$$

# Cylinder object and Left homotopic morphisms

## Definition

- We write $\pi^l(A,X)$ for the set of equivalence classes of $\mathrm{Hom}_{\mathbf{C}}(A,X)$ under the equivalence relation generated by $\overset{l}{\sim}$.
- Let $A$ be a cofibrant object of $C$. Then
  1. $\overset{l}{\sim}$ is an equivalence relation on $\mathrm{Hom}_{\mathbf{C}}(A,X)$ for any $X$ in $\mathbf{C}$. In particular
  $$\mathrm{Hom}_{\mathbf{C}}(A,X)/\overset{l}{\sim} = \pi^l(A,X).$$
  2. For any $p : Y \xrightarrow{\sim} X$ the composition with $p$ induces a bijection
  $$p_* : \pi^l(A,Y) \to \pi^l(A,X), [f] \mapsto [pf].$$

# Homotopy category of a model category

**Definition**

The homotopy category $\mathbf{Ho}(\mathbf{C})$ of a model category $\mathbf{C}$ is the category with same objects as $\mathbf{C}$ and with

$$\mathrm{Hom}_{\mathbf{Ho}(C)}(X,Y) := \mathrm{Hom}_{\mathbf{C}}(RQX, RQY)/ \overset{l}{\sim} = \pi^l(RQX, RQY).$$

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## Remark

There is a functor

$$\gamma : \mathbf{C} \to \mathbf{Ho}(\mathbf{C}) : \begin{cases} X \mapsto X \\ f : X \to Y \mapsto [RQf : RQX \to RQY]. \end{cases}$$

**Theorem**

*The functor* $\gamma : \mathbf{C} \to \mathbf{Ho}(\mathbf{C})$ *is a localization of* $C$ *with respect to the class of weak equivalences* $\mathrm{WE}$.

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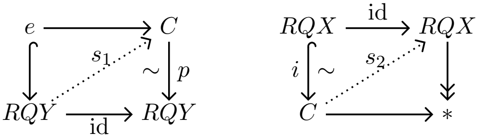


We have $(\gamma f)^{-1} = [s_1 s_2] \in \mathrm{Hom}_{\mathbf{Ho}}(Y, X)$.

# Abelian categories with enough projectives

## Definition

Let $\mathbf{C}$ be a computable abelian category. $\mathbf{C}$ is said to have enough projectives if we can

- decide whether a given object in the category is projective or not.
- compute the (universal) projective lift.
- compute for any object $X$ in $\mathbf{C}$ an epimorphism $P \to X$ from some projective object $P$ in $\mathbf{C}$.

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- Let $R$ is a computable ring, then the category $R$-fpres of (finite) left presentations over $R$ is computable abelian category. If we can compute lifts in $R$-fpres, then it is abelian with enough projectives.

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- Let $Q$ is an acyclic quiver with relations given by ideal $I$. Then the category $\mathrm{fRep}(Q, I)$ of finite representations of $Q$ is computable abelian category with enough projectives. Its projective global dimension in this case is less than the length of a maximal path in $Q$.

# Chain complex category

## Definition

Let $\mathbf{C}$ be an abelian category. The category of chain complexes $\mathbf{Ch}_\bullet(\mathbf{C})$ of $\mathbf{C}$ is the category with objects

$$A_\bullet : \quad \cdots \longleftarrow A_{i-1} \xleftarrow{d_i^A} A_i \xleftarrow{d_{i+1}^A} A_{i+1} \longleftarrow \cdots$$

and morphisms

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A_\bullet : & \cdots \longleftarrow & A_{i-1} & \xleftarrow{d_i^A} & A_i & \xleftarrow{d_{i+1}^A} & A_{i+1} & \longleftarrow \cdots \\
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B_\bullet : & \cdots \longleftarrow & B_{i-1} & \xleftarrow[d_i^B]{} & B_i & \xleftarrow[d_{i+1}^B]{} & B_{i+1} & \longleftarrow \cdots
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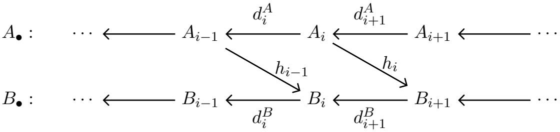
## Theorem

*If $\mathbf{C}$ is computable abelian category, then so is $\mathbf{Ch}_\bullet(\mathbf{C})$.*

# Null-homotopic chain morphisms

Let $A_\bullet, B_\bullet$ be two chain complexes in $\mathbf{Ch}_\bullet(\mathbf{C})$ and let $(h_i : A_i \to B_{i+1})_{i\in\mathbb{Z}}$ be a family of morphisms in $\mathbf{C}$.



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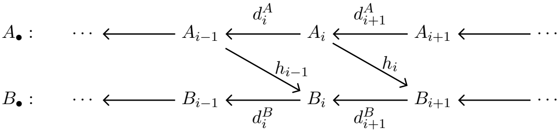


Then the morphisms $(\phi_i := h_i d_{i+1}^B + d_i^A h_{i-1} : A_i \to B_i)_{i\in\mathbb{Z}}$ define a chain morphism $\phi_\bullet : A_\bullet \to B_\bullet$.

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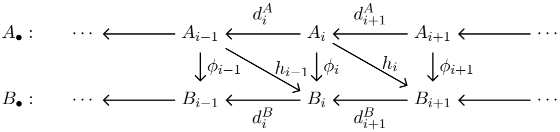


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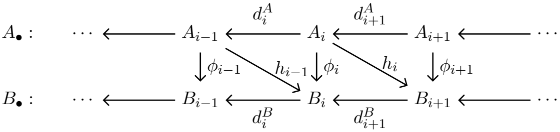


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Such chain morphisms are said to be homotopic to zero or **Null-homotopic** and the family $(h_i)_{i\in\mathbb{Z}}$ is called **Homotopy morphisms** of $\phi_\bullet$.

# Mapping cone

Let $\phi_\bullet : A_\bullet \to B_\bullet$ in $\mathbf{Ch}_\bullet(\mathbf{C})$. We have the following diagram

$$\begin{array}{llllllll}
A_\bullet : & \cdots \longleftarrow & A_{i-1} & \xleftarrow{d_i^A} & A_i & \xleftarrow{d_{i+1}^A} & A_{i+1} & \longleftarrow \cdots \\
 & & \downarrow \phi_{i-1} & & \downarrow \phi_i & & \downarrow \phi_{i+1} & \\
B_\bullet : & \cdots \longleftarrow & B_{i-1} & \xleftarrow{d_i^B} & B_i & \xleftarrow{d_{i+1}^B} & B_{i+1} & \longleftarrow \cdots \\
 & & & & & & & \\
\mathrm{Cone}(\phi_\bullet) : & \cdots \longleftarrow & A_{i-2} \oplus B_{i-1} & \xleftarrow{d_i^{\mathrm{Cone}(\phi_\bullet)}} & A_{i-1} \oplus B_i & \xleftarrow{d_{i+1}^{\mathrm{Cone}(\phi_\bullet)}} & A_i \oplus B_{i+1} & \longleftarrow \cdots
\end{array}$$

where $d_i^{\mathrm{Cone}(\phi_\bullet)}$ is given by the following matrix

$$\begin{bmatrix} d_{i-1}^A & -\phi_{i-1} \\ 0 & d_i^B \end{bmatrix}.$$

# Mapping cone

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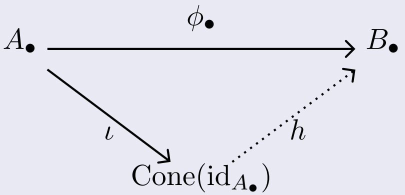
Moreover, we have a natural injection of $B_\bullet$ in $\mathrm{Cone}(\phi_\bullet)$.

## Theorem

*Let* $\phi_\bullet : A_\bullet \to B_\bullet$ *in* $\mathbf{Ch}_\bullet(\mathbf{C})$. *Then* $\phi_\bullet$ *is Null-homotopic iff there is a colift of the natural injection of* $A_\bullet$ *in* $\mathrm{Cone}(\mathrm{id}_{A_\bullet})$ *along* $\phi_\bullet$. *I.e., there is a commutative diagram*



**Theorem**

*Let $\mathbf{C}$ be an abelian category with enough projectives. Define a morphism $\phi_\bullet : A_\bullet \to B_\bullet \in \mathbf{Ch}^{\mathrm{b}}_\bullet(\mathbf{C})$ to be*

- *a weak equivalence if $\phi_\bullet$ is quasi-isomorphism.*
- *a cofibration if for each $k$ the map $\phi_k : A_k \to B_k$ is a monomorphism with projective cokernel object.*
- *a fibration if for each $k$ the map $\phi_k : A_k \to B_k$ is an epimorphism.*

*Then with these choices $\mathbf{Ch}^{\mathrm{b}}_\bullet(\mathbf{C})$ is a model category.*

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In this setting, it can be shown that two morphisms $\phi_\bullet, \psi_\bullet$ in $\mathbf{Ch}^{\mathrm{b}}_\bullet(\mathbf{C})$ are left-homotopic iff $\phi_\bullet - \psi_\bullet$ is Null-homotopic.

**Theorem**

*Let* $\mathbf{C}$ *be a computable abelian category with enough projectives and finite projective global dimension. If lifts and colifts are computable in* $\mathbf{Ch}^b(\mathbf{C})$ *then* $\mathbf{Ho}(\mathbf{Ch}^b(\mathbf{C}))$ *is computable additive (triangulated) category.*

**Theorem**

*Let $\mathbf{C}$ be a computable abelian category with enough projectives and finite projective global dimension. If lifts and colifts are computable in $\mathbf{Ch}^b(\mathbf{C})$ then $\mathbf{Ho}(\mathbf{Ch}^b(\mathbf{C}))$ is computable additive (triangulated) category.*

The colift operation is needed to decide whether a morphism in $\mathbf{Ch}^b(\mathbf{C})$ is null-homotopic or not. The lift operation is needed for the lifting-axiom.

**Example**

If $R$ is computable commutative ring with finite global projective dimension, for example $\mathbb{Z}$ or $k[x_0, \ldots, x_n]$, then $\mathbf{Ho}(\mathbf{Ch}^b(R\text{-fpre}))$ is computable additive category.

**Example**

If $R$ is computable commutative ring with finite global projective dimension, for example $\mathbb{Z}$ or $k[x_0, \ldots, x_n]$, then $\mathbf{Ho}(\mathbf{Ch}^b(R\text{-fpre}))$ is computable additive category.

Here the problem of computing lifts and colifts in $\mathbf{Ch}^b(R\text{-fpre})$ can be reduced to a computation of a lifts in $R$-fpre.

**Example**

Let $Q$ is an acyclic quiver with relation given by ideal $I$. Then $\mathbf{Ho}(\mathbf{Ch}^b(\mathrm{fRep}(Q,I)))$ is computable additive category.

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Any bounded chain complex of quiver representations can be considered as a representation of some other bigger quiver with relations, say $(Q', I')$. Hence the problem of computing lifts and colifts in $\mathbf{Ch}^b(\mathrm{fRep}(Q, I))$ can be reduced to computation of lifts and colifts in $\mathrm{fRep}(Q', I')$.

# $\mathbf{Ho}(\mathbf{Ch}^b_\bullet(k[x_0,\ldots,x_n]\text{-fpres}))$

The Gap/Cap demo can be found at
`https://github.com/kamalsaleh/ModelCategories`
The following packages are required

- `CAP`
- `RingsForHomalg`
- `ToolsForHomalg`
- `ModulePresentationsForCAP`
- `InfiniteLists`
- `ComplexesForCAP`
- `TriangulatedCategoriesForCAP`
- `ModelCategories`

# $\mathbf{Ho}(\mathbf{Ch}^b(\mathrm{fRep}(Q, I)))$ for $Q$ the Beilinson quiver

The Gap/Cap demo can be found at
`https://github.com/kamalsaleh/ModelCategories`
The following packages are required

- `CAP`
- `QPA2`
- `InfiniteLists`
- `ComplexesForCAP`
- `TriangulatedCategoriesForCAP`
- `ModelCategories`

1. Homotopy theories and model categories, *W. G. Dwyer and J. Spalinski*
2. Model categories, *M. Hovey*
3. An axiomatic setup for algorithmic homological algebra and an alternative approach to localization, *M. Barakat, M. Lange-Hegermann*
4. CAP, *S. Gutsche, S. Posur, M. Barakat, Ø. Skartsæterhagen*

*Thanks*