CddInterface

Gap interface to Cdd package

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Chapter 1

Introduction

1.1 Why CddInterface

We know that every convex polyhedron has two representations, one as the intersection of finite half-spaces and the other as Minkowski sum of the convex hull of finite points and the nonnegative hull of finite directions. These are called H-representation and V-representation, respectively. CddInterface is a gap interface to the C package `cddlib` which among other things can translate between these two representations.

1.2 H-representation and V-representation of polyhedra

Let us start by introducing the H-representation. Let *A* be *m x d* matrix and let *b* be a column *m*-vector. The H-representation of the polyhedron defined by the system `b+Ax >= 0` of *m* inequalities and *d* variables *x= (x_1,...,x_d)* is as follows:

Code
```
H-representation
linearity t, [i_1, i_2, ...,i_t]
begin
  m x (d+1) numbertype
  b  A
end
```

The linearity line is added when we want to specify that some rows of the system $b+Ax$ are equalities. That is, $k \in \{i_1, i_2, \ldots, i_t\}$ means that the row k of the system $b+Ax$ is specified to be equality.

For example, the H-representation of the polyhedron defined by the following system:

$4-3x_1+6x_2-5x_4=0, 1+2x_1-2x_2-7x_3 \geq 0, -3x_2+5x_4=0;$

is the following:

Code
```
H-representation
linearity 2, [1, 3]
begin
3 x 5 rational
  4 -3  6  0 -5
  1  2 -2 -7  0
```

```
  0  0 -3  0  5
end
```

Next we define Polyhedra V-format. Let `P` be represented by `n` gerating points and `s` generating directions (rays) as

$$P = \mathrm{conv}(v_1, \ldots, v_n) + \mathrm{nonneg}(r_{n+1}, \ldots, r_{n+s}).$$

Then the Polyhedra V-format is for `P` is:

Code

```
V-representation
linearity t, [i_1, i_2,...,i_t]
begin
(n+s) x (d+1) numbertype
  1  v_1
  :    :
  1  v_n
  0  r_{n+1}
  :    :
  0  r_{n+s}
end
```

In the above format the generating points and generating rays may appear mixed in arbitrary order. Linearity for V-representation specifies a subset of generators whose coefficients are relaxed to be free. That is, $k \in \{i_1, i_2, ..., i_t\}$ specifies that the k-th generator is specified to be free. This means for each such a ray r_k , the line generated by r_k is in the polyhedron, and for each such a vertex v_k , its coefficient is no longer nonnegative but still the coefficients for all v_i's must sum up to one.

For example the V-representation of the polyhedron defined as

$$P := \mathrm{conv}((2,3),(-2,-3),(-1,2)) + \mathrm{nonneg}(\ (1,2),(-1,-2),(1,1)\ )$$

Code

```
V-representation
linearity 2, [ 1, 3 ]
begin
 4 x 3 rational
 1   2   3
 1  -1   2
 0   1   2
 0   1   1
end
```

Chapter 2

Creating polyhedra and their Operations

2.1 Creating a polyhedron

2.1.1 Cdd_PolyhedronByInequalities

▷ `Cdd_PolyhedronByInequalities(ineq[, linearities_list])` (function)

Returns: a CddPolyhedron

The function takes a list in which every entry represents an inequality (or equality). In case we want some entries to represent equalities we should refer in a second list to their indices.

Example

```
gap> A:= Cdd_PolyhedronByInequalities( [ [ 0, 1, 0 ], [ 0, 1, -1 ] ] );
<Polyhedron given by its H-representation>
gap> Display( A );
H-representation
begin
   2 X 3  rational

   0   1   0
   0   1  -1
end
gap> B:= Cdd_PolyhedronByInequalities( [ [ 0, 1, 0 ], [ 0, 1, -1 ] ], [ 2 ] );
<Polyhedron given by its H-representation>
gap> Display( B );
H-representation
linearity 1, [ 2 ]
begin
   2 X 3  rational

   0   1   0
   0   1  -1
end
```

2.1.2 Cdd_PolyhedronByGenerators

▷ `Cdd_PolyhedronByGenerators(genes[, linearities_list])` (function)

Returns: a CddPolyhedron

The function takes a list in which every entry represents a vertex in the ambient vector space. In case we want some vertices to be free (the vertex and its negative belong to the polyhedron) we should refer in a second list to their indices .

Example
```
gap> A:= Cdd_PolyhedronByGenerators( [ [ 0, 1, 3 ], [ 1, 4, 5 ] ] );
<Polyhedron given by its V-representation>
gap> Display( A );
V-representation
begin
   2 X 3  rational

   0  1  3
   1  4  5
end
gap> B:= Cdd_PolyhedronByGenerators( [ [ 0, 1, 3 ] ], [ 1 ] );
<Polyhedron given by its V-representation>
gap> Display( B );
V-representation
linearity 1, [ 1 ]
begin
   1 X 3  rational

   0  1  3
end
```

2.2 Some operations on a polyhedron

2.2.1 Cdd_FourierProjection (for IsCddPolyhedron, IsInt)

▷ Cdd_FourierProjection(*P*, *i*) (operation)

Returns: a CddPolyhedron

The function returns the Fourier projection of the polyhedron in the subspace $(O,x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)$ after applying the Fourier elemination algorithm to get rid of the variable x_i.

To illustrate this projection, Let $P = \mathrm{conv}((1,2),(4,5))$ in $\mathbb{Q}^2$.
To find its projection on the subspace (O,x_1), we apply the Fourier elemination to get rid of x_2

Example
```
gap> P := Cdd_PolyhedronByGenerators( [ [ 1, 1, 2 ], [ 1, 4, 5 ] ] );
<Polyhedron given by its V-representation>
gap> H := Cdd_H_Rep( P );
<Polyhedron given by its H-representation>
gap> Display( H );
H-representation
linearity 1, [ 3 ]
begin
   3 X 3  rational

    4  -1   0
   -1   1   0
   -1  -1   1
```

```
end
gap> P_x1 := Cdd_FourierProjection( H, 2);
<Polyhedron given by its H-representation>
gap> Display( P_x1 );
H-representation
linearity 1, [ 3 ]
begin
   3 X 3  rational

    4  -1   0
   -1   1   0
    0   0   1
end
gap> Display( Cdd_V_Rep( P_x1 ) );
V-representation
begin
   2 X 3  rational

   1  1  0
   1  4  0
end
```

Let again $Q = Conv((2,3,4),(2,4,5)) + nonneg((1,1,1))$, and let us compute its projection on (O,x_2,x_3)

Example

```
gap> Q := Cdd_PolyhedronByGenerators( [ [ 1, 2, 3, 4 ],[ 1, 2, 4, 5 ], [ 0, 1, 1, 1 ] ] );
<Polyhedron given by its V-representation>
gap> R := Cdd_H_Rep( Q );
<Polyhedron given by its H-representation>
gap> Display( R );
H-representation
linearity 1, [ 4 ]
begin
   4 X 4  rational

    2   1  -1   0
   -2   1   0   0
   -1  -1   1   0
   -1   0  -1   1
end
gap> P_x2_x3 := Cdd_FourierProjection( R, 1);
<Polyhedron given by its H-representation>
gap> Display( P_x2_x3 );
H-representation
linearity 2, [ 1, 3 ]
begin
   3 X 4  rational

   -1   0  -1   1
   -3   0   1   0
    0   1   0   0
end
```

```
gap> Display( Cdd_V_Rep( last ) ) ;
V-representation
begin
   2 X 4  rational

   0  0  1  1
   1  0  3  4
end
```

2.3 Some operations on two polyhedrons

2.3.1 Cdd_IsContained (for IsCddPolyhedron, IsCddPolyhedron)

▷ `Cdd_IsContained(P1, P2)` (operation)

Returns: `true` or `false`

The function returns `true` if P_1 is contained in P_2, otherwise returns `false`.

Example

```
gap> A := Cdd_PolyhedronByInequalities( [ [ 10, -1, 1, 0 ],
> [ -24, 9, 2, 0 ], [ 1, 1, -1, 0 ], [ -23, -12, 1, 11 ] ], [ 4 ] );
<Polyhedron given by its H-representation>
gap> B := Cdd_PolyhedronByInequalities( [ [ 1, 0, 0, 0 ],
> [ -4, 1, 0, 0 ], [ 10, -1, 1, 0 ], [ -3, -1, 0, 1 ] ], [ 3, 4 ] );
<Polyhedron given by its H-representation>
gap> Cdd_IsContained( B, A );
true
gap> Display( Cdd_V_Rep( A ) );
V-representation
begin
   3 X 4  rational

   1   2   3   4
   1   4  -6   7
   0   1   1   1
end
gap> Display( Cdd_V_Rep( B ) );
V-representation
begin
   2 X 4  rational

   1   4  -6   7
   0   1   1   1
end
```

2.3.2 Cdd_Intersection (for IsCddPolyhedron, IsCddPolyhedron)

▷ `Cdd_Intersection(P1, P2)` (operation)

Returns: a CddPolyhedron

The function returns the intersection of P_1 and P_2

Example
```
gap> A := Cdd_PolyhedronByInequalities( [ [ 3, 4, 5 ] ], [ 1 ] );;
gap> B := Cdd_PolyhedronByInequalities( [ [ 9, 7, 2 ] ], [ 1 ] );;
gap> C := Cdd_Intersection( A, B );;
gap> Display( Cdd_V_Rep( A ) );
V-representation
linearity 1, [ 2 ]
begin
   2 X 3  rational

   1  -3/4     0
   0    -5     4
end
gap> Display( Cdd_V_Rep( B ) );
V-representation
linearity 1, [ 2 ]
begin
   2 X 3  rational

   1  -9/7     0
   0    -2     7
end
gap> Display( Cdd_V_Rep( C ) );
V-representation
begin
   1 X 3  rational

   1  -13/9    5/9
end
```

2.3.3 \+ (for IsCddPolyhedron, IsCddPolyhedron)

▷ \+(*P1*, *P2*) (operation)

Returns: a CddPolyhedron

The function returns the Minkuwski sum of P_1 and P_2.

Example
```
gap> P := Cdd_PolyhedronByGenerators( [ [ 1, 2, 5 ], [ 0, 1, 2 ] ] );
< Polyhedron given by its V-representation >
gap> Q := Cdd_PolyhedronByGenerators( [ [ 1, 4, 6 ], [ 1, 3, 7 ], [ 0, 3, 1 ] ] );
< Polyhedron given by its V-representation >
gap> S := P+Q;
< Polyhedron given by its H-representation >
gap> V := Cdd_V_Rep( S );
< Polyhedron given by its V-representation >
gap> Display( V );
V-representation
begin
   4 X 3  rational

   0   3   1
   1   6  11
   1   5  12
```

```
    0   1   2
end
gap> Cdd_GeneratingVertices( P );
[ [ 2, 5 ] ]
gap> Cdd_GeneratingVertices( Q );
[ [ 3, 7 ], [ 4, 6 ] ]
gap> Cdd_GeneratingVertices( S );
[ [ 5, 12 ], [ 6, 11 ] ]
gap> Cdd_GeneratingRays( P );
[ [ 1, 2 ] ]
gap> Cdd_GeneratingRays( Q );
[ [ 3, 1 ] ]
gap> Cdd_GeneratingRays( S );
[ [ 1, 2 ], [ 3, 1 ] ]
```

Chapter 3

Linear Programs

3.1 Creating and solving linear programs

3.1.1 Cdd_LinearProgram (for IsCddPolyhedron, IsString, IsList)

▷ Cdd_LinearProgram(*P, str, obj*) (operation)

Returns: a *CddLinearProgram* Object

The function takes three variables. The first is a polyhedron *poly*, the second *str* should be "max" or "min" and the third *obj* is the objective function.

3.1.2 Cdd_SolveLinearProgram (for IsCddLinearProgram)

▷ Cdd_SolveLinearProgram(*lp*) (operation)

Returns: a list if the program is optimal, otherwise returns the value 0

The function takes a linear program. If the program is optimal, the function returns a list of two entries, the solution vector and the optimal value of the objective, otherwise it returns `fail`.

To illustrate the using of these functions, let us solve the linear program given by:

$$\textbf{Maximize } P(x,y) = 1 - 2x + 5y, \text{ with}$$

$$100 \leq x \leq 200, 80 \leq y \leq 170, y \geq -x + 200.$$

We bring the inequalities to the form $b + AX \geq 0$ and get:

$$-100 + x \geq 0, 200 - x \geq 0, -80 + y \geq 0, 170 - y \geq 0, -200 + x + y \geq 0.$$

Example

```
gap> A:= Cdd_PolyhedronByInequalities( [ [ -100, 1, 0 ], [ 200, -1, 0 ],
> [ -80, 0, 1 ], [ 170, 0, -1 ], [ -200, 1, 1 ] ] );
<Polyhedron given by its H-representation>
gap> lp1:= Cdd_LinearProgram( A, "max", [1, -2, 5 ] );
<Linear program>
gap> Display( lp1 );
Linear program given by:
H-representation
begin
   5 X 3  rational
```

```
   -100     1     0
    200    -1     0
    -80     0     1
    170     0    -1
   -200     1     1
end
max  [ 1, -2, 5 ]
gap> Cdd_SolveLinearProgram( lp1 );
[ [ 100, 170 ], 651 ]
gap> lp2:= Cdd_LinearProgram( A, "min", [ 1, -2, 5 ] );
<Linear program>
gap> Display( lp2 );
Linear program given by:
H-representation
begin
   5 X 3  rational

   -100     1     0
    200    -1     0
    -80     0     1
    170     0    -1
   -200     1     1
end
min  [ 1, -2, 5 ]
gap> Cdd_SolveLinearProgram( lp2 );
[ [ 200, 80 ], 1 ]
gap> B:= Cdd_V_Rep( A );
<Polyhedron given by its V-representation>
gap> Display( B );
V-representation
begin
   5 X 3  rational

   1  100  170
   1  100  100
   1  120   80
   1  200   80
   1  200  170
end
```

So the optimal solution for `lp1` is $(x = 100, y = 170)$ with optimal value $p = 1 - 2(100) + 5(170) = 651$ and for `lp2` is $(x = 200, y = 80)$ with optimal value $p = 1 - 2(200) + 5(80) = 1$.

Chapter 4

Attributes and properties

4.1 Attributes and properties of polyhedron

4.1.1 Cdd_Canonicalize (for IsCddPolyhedron)

▷ `Cdd_Canonicalize(P)` (attribute)

Returns: a CddPolyhedron

The function takes a polyhedron and reduces its defining inequalities (generators set) by deleting all redundant inequalities (generators).

Example
```
gap> A:= Cdd_PolyhedronByInequalities( [ [ 0, 2, 6 ], [ 0, 1, 3 ], [1, 4, 10 ] ] );
<Polyhedron given by its H-representation>
gap> B:= Cdd_Canonicalize( A );
<Polyhedron given by its H-representation>
gap> Display( B );
H-representation
begin
   2 X 3  rational

   0   1   3
   1   4  10
end
```

4.1.2 Cdd_V_Rep (for IsCddPolyhedron)

▷ `Cdd_V_Rep(P)` (attribute)

Returns: a CddPolyhedron

The function takes a polyhedron and returns its reduced V-representation.

4.1.3 Cdd_H_Rep (for IsCddPolyhedron)

▷ `Cdd_H_Rep(P)` (attribute)

Returns: a CddPolyhedron

The function takes a polyhedron and returns its reduced H-representation.

Example
```
gap> A:= Cdd_PolyhedronByInequalities( [ [ 0, 1, 1 ], [ 0, 5, 5 ] ] );
<Polyhedron given by its H-representation>
```

```
gap> B:= Cdd_V_Rep( A );
<Polyhedron given by its V-representation>
gap> Display( B );
V-representation
linearity 1, [ 2 ]
begin
   2 X 3  rational

   0   1   0
   0  -1   1
end
gap> C:= Cdd_H_Rep( B );
<Polyhedron given by its H-representation>
gap> Display( C );
H-representation
begin
   1 X 3  rational

   0  1  1
end
gap> D:= Cdd_PolyhedronByInequalities( [ [ 0, 1, 1, 34, 22, 43 ],
> [ 11, 2, 2, 54, 53, 221 ], [33, 23, 45, 2, 40, 11 ] ] );
<Polyhedron given by its H-representation>
gap> Cdd_V_Rep( D );
<Polyhedron given by its V-representation>
gap> Display( last );
V-representation
linearity 2, [ 5, 6 ]
begin
   6 X 6  rational

   1  -743/14   369/14    11/14        0        0
   0    -1213      619       22        0        0
   0       -1        1        0        0        0
   0      764     -390      -11        0        0
   0   -13526     6772       99      154        0
   0  -116608    59496     1485        0      154
end
```

4.1.4 Cdd_AmbientSpaceDimension (for IsCddPolyhedron)

▷ Cdd_AmbientSpaceDimension(*P*) (attribute)

Returns: The dimension of the ambient space of the polyhedron(i.e., the space that contains *P*).

4.1.5 Cdd_Dimension (for IsCddPolyhedron)

▷ Cdd_Dimension(*P*) (attribute)

Returns: The dimension of the polyhedron, where the dimension, $\dim(P)$, of a polyhedron P is the maximum number of affinely independent points in P minus 1.

4.1.6 Cdd_GeneratingVertices (for IsCddPolyhedron)

▷ `Cdd_GeneratingVertices(P)` (attribute)

Returns: The reduced generating vertices of the polyhedron

4.1.7 Cdd_GeneratingRays (for IsCddPolyhedron)

▷ `Cdd_GeneratingRays(P)` (attribute)

Returns: list

The output is the reduced generating rays of the polyhedron

4.1.8 Cdd_Equalities (for IsCddPolyhedron)

▷ `Cdd_Equalities(P)` (attribute)

Returns: a list

The output is the reduced equalities of the polyhedron.

4.1.9 Cdd_Inequalities (for IsCddPolyhedron)

▷ `Cdd_Inequalities(P)` (attribute)

The output is the reduced inequalities of the polyhedron.

4.1.10 Cdd_InteriorPoint (for IsCddPolyhedron)

▷ `Cdd_InteriorPoint(P)` (attribute)

Returns: a list

The output is an interior point in the polyhedron

4.1.11 Cdd_Faces (for IsCddPolyhedron)

▷ `Cdd_Faces(P)` (attribute)

Returns: a list

This function takes a H-represented polyhedron P and returns a list. Every entry in this list is a again a list, contains the dimension and linearity of the face defined as a polyhedron over the same system of inequalities.

4.1.12 Cdd_FacesWithFixedDimension (for IsCddPolyhedron, IsInt)

▷ `Cdd_FacesWithFixedDimension(P, d)` (operation)

Returns: a list

This function takes a H-represented polyhedron P and a positive integer d. The output is a list. Every entry in this list is the linearity of an d- dimensional face of P defined as a polyhedron over the same system of inequalities.

4.1.13 Cdd_FacesWithInteriorPoints (for IsCddPolyhedron)

▷ `Cdd_FacesWithInteriorPoints(P)` (attribute)

Returns: a list

This function takes a H-represented polyhedron P and returns a list. Every entry in this list is a again a list, contains the dimension, linearity of the face defined as a polyhedron over the same system of inequalities and an interior point in the face.

4.1.14 Cdd_FacesWithFixedDimensionAndInteriorPoints (for IsCddPolyhedron, IsInt)

▷ `Cdd_FacesWithFixedDimensionAndInteriorPoints(P, d)` (operation)

Returns: a list

This function takes a H-represented polyhedron P and a positive integer d. The output is a list. Every entry in this list is a again a list, contains the linearity of the face defined as a polyhedron over the same system of inequalities and an interior point in this face.

4.1.15 Cdd_Facets (for IsCddPolyhedron)

▷ `Cdd_Facets(P)` (attribute)

Returns: a list

This function takes a H-represented polyhedron P and returns a list. Every entry in this is the linearity of a facet defined as a polyhedron over the same system of inequalities.

4.1.16 Cdd_Lines (for IsCddPolyhedron)

▷ `Cdd_Lines(P)` (attribute)

Returns: a list

This function takes a H-represented polyhedron P and returns a list. Every entry in this is the linearity of a ray (1-dimensional face) defined as a polyhedron over the same system of inequalities.

4.1.17 Cdd_Vertices (for IsCddPolyhedron)

▷ `Cdd_Vertices(P)` (attribute)

Returns: a list

This function takes a H-represented polyhedron P and returns a list. Every entry in this list is the linearity of a vertex defined as a polyhedron over the same system of inequalities.

4.1.18 Cdd_IsEmpty (for IsCddPolyhedron)

▷ `Cdd_IsEmpty(P)` (property)

Returns: true or false

The output is `true` if the polyhedron is empty and `false` otherwise

4.1.19 Cdd_IsCone (for IsCddPolyhedron)

▷ `Cdd_IsCone(P)` (property)

Returns: true or false

The output is `true` if the polyhedron is cone and `false` otherwise

4.1.20 Cdd_IsPointed (for IsCddPolyhedron)

▷ Cdd_IsPointed(*P*) (property)

Returns: true or false

The output is `true` if the polyhedron is pointed and `false` otherwise

Example
```
gap> poly:= Cdd_PolyhedronByInequalities( [ [ 1, 3, 4, 5, 7 ], [ 1, 3, 5, 12, 34 ],
> [ 9, 3, 0, 2, 13 ]  ], [ 1 ] );
<Polyhedron given by its H-representation>
gap> Cdd_InteriorPoint( poly );
[ -194/75, 46/25, -3/25, 0 ]
gap> Cdd_FacesWithInteriorPoints( poly );
[ [ 3, [ 1 ], [ -194/75, 46/25, -3/25, 0 ] ], [ 2, [ 1, 2 ],
[ -62/25, 49/25, -7/25, 0 ] ], [ 1, [ 1, 2, 3 ],
[ -209/75, 56/25, -8/25, 0 ] ], [ 2, [ 1, 3 ], [ -217/75, 53/25, -4/25, 0 ] ] ]
gap> Cdd_Dimension( poly );
3
gap> Cdd_IsPointed( poly );
false
gap> Cdd_IsEmpty( poly );
false
gap> Cdd_Faces( poly );
[ [ 3, [ 1 ] ], [ 2, [ 1, 2 ] ], [ 1, [ 1, 2, 3 ] ], [ 2, [ 1, 3  ] ] ]
gap> poly1 := Cdd_ExtendLinearity( poly, [ 1, 2, 3 ] );
<Polyhedron given by its H-representation>
gap> Display( poly1 );
H-representation
linearity 3, [ 1, 2, 3 ]
begin
   3 X 5  rational

   1   3   4   5   7
   1   3   5  12  34
   9   3   0   2  13
end
gap> Cdd_Dimension( poly1 );
1
gap> Cdd_Facets( poly );
[ [ 1, 2 ], [ 1, 3 ] ]
gap> Cdd_GeneratingVertices( poly );
[ [ -209/75, 56/25, -8/25, 0 ] ]
gap> Cdd_GeneratingRays( poly );
[ [ -97, 369, -342, 75 ], [ -8, -9, 12, 0 ],
[ 23, -21, 3, 0 ], [ 97, -369, 342, -75 ] ]
gap> Cdd_Inequalities( poly );
[ [ 1, 3, 5, 12, 34 ], [ 9, 3, 0, 2, 13 ] ]
gap> Cdd_Equalities( poly );
[ [ 1, 3, 4, 5, 7 ] ]
gap> P := Cdd_FourierProjection( poly, 2);
<Polyhedron given by its H-representation>
gap> Display( P );
H-representation
linearity 1, [ 3 ]
```

```
begin
   3 X 5  rational

    9    3    0    2   13
   -1   -3    0   23  101
    0    0    1    0    0
end
```

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